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Brownian motion in AdS/CFT

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Abstract: We study Brownian motion and the associated Langevin equation in AdS/CFT. The Brownian particle is realized in the bulk spacetime as a probe fundamental string in an asymptotically AdS black hole background, stretching between the AdS boundary and the horizon. The modes on the string are excited by the thermal black hole environment and consequently the string endpoint at the boundary undergoes an erratic motion, which is identified with an external quark in the boundary CFT exhibiting Brownian motion. Semiclassically, the modes on the string are thermally excited due to Hawking radiation, which translates into the random force appearing in the boundary Langevin equation, while the friction in the Langevin equation corresponds to the excitation on the string being absorbed by the black hole. We give a bulk proof of the fluctuation-dissipation theorem relating the random force and friction. This work can be regarded as a step toward understanding the quantum microphysics underlying the fluid-gravity correspondence. We also initiate a study of the properties of the effective membrane or stretched horizon picture of black holes using our bulk description of Brownian motion.

Keywords: Black Holes in String Theory, AdS-CFT Correspondence, Hadronic Colliders, Boundary Quantum Field Theory

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1 Introduction

One of the interesting problems in statistical mechanics concerns the understanding of the origin of macroscopic dissipation and the approach to thermal equilibrium from microscopical point of view. Conventionally, given a statistical system in the thermodynamic or hydrodynamic limit, we imagine the collisions between the microscopic constituents of our system as being responsible for both of these macroscopic phenomena. This kinetic theory based picture is firmly anchored on the basic idea of Brownian motion — in 1827, the botanist Robert Brown observed [1] under a microscope that tiny pollen particles suspended in water undergo incessant irregular motion, which became known as the Brownian motion.[1] As is well-known now, this peculiar motion is due to collisions with the fluid particles in random thermal motion. Therefore, any particle immersed in fluid at finite temperature exhibits such Brownian motion, from a small pendulum suspended in a dilute gas [6] to a heavy particle in quark-gluon plasma. This universal phenomenon suggests that the interaction with microscopic constituents is responsible for dissipation and thermalization on macroscopic scales.

Since its advent, the holographic AdS/CFT correspondence [7–10] has been exploited to study the physics of non-Abelian quark-gluon plasmas at finite temperature from bulk gravitational physics, and *vice versa*. The dual gravitational description of strongly coupled gauge theories provides an efficient way to study the thermodynamic properties and the phase structure of the gauge theory. More recently, it has become clear that one can also exploit the gravitational description to understand the hydrodynamic regime of the quark-gluon plasma, as was originally proposed in [11] and has been significantly developed afterwards (see [12] and references therein for earlier work on hydrodynamics in the AdS/CFT context). Namely, the long-wavelength physics described by a hydrodynamical Navier-Stokes equation on the boundary side is holographically dual to the long-wavelength fluctuation of the horizons of asymptotically AdS black hole spacetimes on the gravitational side. This correspondence allows for a detailed quantitative study of the plasma from the bulk, and *vice versa*. It is thus a natural question to ask whether one can obtain a holographic description of Brownian motion, which is one step towards the microphysics underlying thermodynamics and hydrodynamics. The aim of this paper is to answer this question in the affirmative.

One intrinsic reason to be interested in Brownian motion within a holographic setting is to better understand the microscopic origin of the thermodynamic properties of black holes. It has become clear from the formulation of the AdS/CFT correspondence that one has an in-principle solution to the problem of quantum dynamics of black holes: we only need to solve the problem phrased in terms of the dual field theory variables. However, it is fair to say that a concrete quantitative understanding of the physics in these contexts is still lacking. One of the most useful playgrounds for understanding the quantum behavior of black holes has been the arena of supersymmetric black holes [13, 14]. Here we not only understand in many cases the microscopic origin of black hole entropy, but also in a number of cases have a bulk picture of the nature of the microscopic states making up the black

[1] Classic reviews on Brownian motion are [2–4]. For a more complete list of references, see e.g. [5].

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hole degeneracy. In fact, from these various analyses, there emerges a rather intriguing picture of a quantum black hole — the black hole microstates form a sort of spacetime foam that replaces the region inside the horizon. Any single microstate is horizon-free, but the typical microstates are expected to exhibit the characteristic features of black hole spacetimes, which has been confirmed explicitly for some concrete systems; see [15–18] for reviews. Given this state of affairs one might probe these microstates beyond equilibrium thermodynamics and ask how the ensemble of them leads to dissipation and thermalization seen in a thermal medium. Understanding the description of Brownian motion seems then a natural step towards getting a handle on the problem.

Conversely, as mentioned above, the AdS/CFT correspondence has been immensely useful in understanding many qualitative (and sometimes quantitative) features of quark-gluon plasmas. The famous lower bound on the ratio of shear-viscosity to entropy density for relativistic hydrodynamic systems, $\eta/s \geq 1/4\pi$ [19] (see also a review [12]), has certainly played an important role in obtaining a quantitative understanding of the dynamics of the quark-gluon plasma produced at RHIC. Furthermore, studies of the motion of quarks, mesons, and baryons in the quark-gluon plasma have been carried out in the holographic framework starting with the seminal papers [20–27], by considering the dynamics of probe strings and D-branes in asymptotically AdS black hole spacetime — for a sample of recent reviews on the subject, see [28]. The general philosophy in these discussions was to use the probe dynamics to extract the rates of energy loss and transverse momentum broadening in the medium, which bear direct relevance to the physical problem of motion of quarks and mesons in the quark-gluon plasma. In such computations, the motion of an external quark in the quark-gluon plasma is assumed to be described by a relativistic Langevin equation [29]. In the most basic form, the Langevin equation is parametrized by two constants: the friction (drag force) coefficient γ and the magnitude of the random force κ.[2] Furthermore, the random force is assumed to be white noise. By using the AdS/CFT realization of external quarks, refs. [20, 22] determined the friction coefficient γ, while refs. [24, 25, 27] computed the random force κ.[3]

Therefore, one can say that the most basic data of the Langevin equation describing Brownian motion in the CFT plasma are already available. However, rather than taking such approaches which are phenomenological in some sense, one could study more fundamental aspects of Brownian motion in the AdS/CFT context. For example, in the first place, why does an external quark exhibit Brownian motion, and why is the motion described by a Langevin equation? While the domain of validity of the Langevin equation is clear from the previous results on the drag force, can we identify the origins of the Brownian motion approximation from the bulk gravitational description? What is the bulk meaning

[2] More precisely, in the relativistic case, the random force has different magnitudes κ_L and κ_T in the directions transverse and longitudinal to the momentum p. In the non-relativistic limit $p \to 0$, they are equal: $\kappa_L = \kappa_T$. The parameters γ and κ_L are related to each other by the Einstein relation, under the assumption that the Langevin dynamics holds and gives the Jüttner distribution $e^{-\beta E}$. On the other hand, κ_T is an independent parameter [25, 29].

[3] More precisely, [20, 22] computed γ and [25, 27] computed κ_T, both in the relativistic case (the computation of [24] was nonrelativistic). The longitudinal component κ_L does not have to be computed independently, since it is related to γ by the Einstein relation. See also footnote 2.

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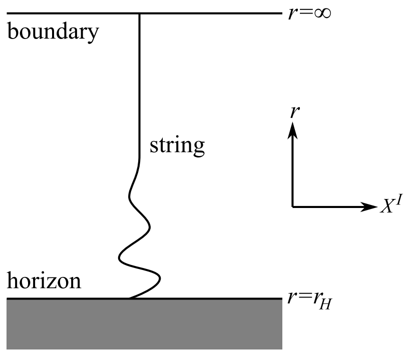

Figure 1. The bulk dual of a Brownian particle: a fundamental string hanging from the boundary of the AdS space and dipping into the horizon. The AdS black hole environment excites the modes on the string and, as a result, the string endpoint at infinity moves randomly, corresponding to the Brownian motion on the boundary.

of the fluctuation-dissipation theorem relating γ and κ? The main purpose of the current paper is to elucidate the AdS/CFT physics of Brownian motion, by addressing such questions. For example, the computation of the random force in [24, 25, 27] using the GKPW prescription [8, 9] does not explain what the bulk counterpart of the random force is. We will see that it corresponds to a version of Hawking radiation in the bulk.[4]

Since we want to model Brownian motion, we need a gravitational analog of a particle immersed in a thermal medium. In the boundary field theory a natural particle is a test quark of large but finite mass immersed in the quark-gluon plasma. This is realized in the dual gravitational picture by introducing a fundamental string in the Schwarzschild-AdS background. The endpoint of the string at the boundary then corresponds to the test quark which undergoes Brownian motion; see figure 1.

We will use this simple picture of a probe fundamental string in a black hole background to "derive" the Brownian motion which the string endpoint on the boundary undergoes.[5] The basic idea is to quantize the fluctuations of the string world-sheet about a classical solution, which in the situations of interest corresponds to a straight string hanging down from the boundary. Since the bulk geometry has an event horizon, the induced metric on the string world-sheet also corresponds to a black hole geometry and the problem of studying fluctuations reduces to the dynamics of two dimensional quantum fields in curved spacetime. By quantizing the fluctuations we relate the quantum modes of the string to the boundary endpoint. This mapping in principle allows one to use the correlation functions of the position of the string endpoint on the boundary to recover the excitation spectrum of the string world-sheet. Assuming the validity of the semiclassical approximation, we can then relate the thermal physics of the Hawking radiation to the Brownian motion of the string endpoint and derive the Langevin equation for the boundary dynamics.

[4]For an earlier discussion of the relation between Hawking radiation and diffusion the context of AdS/QCD, see [30].

[5]A preliminary discussion of the fluctuations of a fundamental string in an asymptotically AdS black hole background can be found in [31].

The organization of the rest of the paper is as follows. In section 2, we set the stage for our discussion by first reviewing the Langevin equation describing Brownian motion in the field theory context. We then turn to a holographic realization of Brownian motion in terms of the dynamics of a probe fundamental string stretching from the boundary to the horizon of an asymptotically AdS black hole geometry in d dimensions. We write down the explicit relation between the boundary and bulk quantities associated with the holographic Brownian motion. This boundary-bulk relation can be explicitly worked out at the semiclassical level, which we turn to in section 3, focussing on the simple case of three dimensional spacetimes. There, we assume that modes on the string are thermally excited due to Hawking radiation and we derive the Langevin dynamics exhibited by the boundary Brownian particle. The friction and the random force appearing in the Langevin equation are related to each other by the fluctuation-dissipation theorem. In section 4, we study this theorem from the bulk viewpoint and give a bulk proof of it in the general case. In section 5, we generalize the discussion in section 3 for $d = 3$ to general dimensions. Despite being unable to quantize the modes on the string analytically in this case we nevertheless show that at small frequencies we recover the Langevin equation. In section 6, we study whether the bulk Brownian motion of the fundamental string can be interpreted as being caused by a suitable movement of the string endpoint on the horizon, the idea being that the endpoint is randomly excited by the stringy gas living on a membrane just outside the horizon, much as in the spirit of the membrane paradigm. We also provide a preliminary discussion of how to use our setup to study microscopic properties of the stretched horizon. Ultimately, we would like to directly probe properties of the quasi-particles that make up the stringy gas living at the stretched horizon, but that is beyond the scope of the present paper. Section 7 is devoted to a discussion. Some of the relevant technical details are collected in the appendices.

2 Holographic Brownian motion

To set the stage for our discussion we begin with a brief review of the Langevin dynamics that describes the Brownian motion. This discussion will be the field theoretic, or boundary, side of the story in the AdS/CFT context. Turning to the corresponding bulk description, we will then describe how one can set up the problem of studying the motion of a Brownian particle in a thermal medium in terms of a probe string in an asymptotically Schwarzschild-AdS black hole background.

2.1 Brownian motion and Langevin dynamics

Let us begin with the Langevin equation, which is the simplest model describing a non-relativistic Brownian particle of mass m in one spatial dimension:

$$\dot{p}(t) = -\gamma_0\, p(t) + R(t), \tag{2.1}$$

where $p = m\dot{x}$ is the (non-relativistic) momentum of the Brownian particle at position x, and $\dot{} \equiv d/dt$. The two terms on the right hand side of (2.1) correspond to friction and a random force, respectively, and γ_0 is a constant called the friction coefficient. One can think of the particle as losing energy to the medium due to the friction term and concurrently getting a random kick from the thermal bath modeled by the random force, which we assume to be white noise with the following average:

$$\langle R(t)\rangle = 0, \qquad \langle R(t)R(t')\rangle = \kappa_0\, \delta(t-t'), \tag{2.2}$$

where κ_0 is a constant. The separation of the force into frictional and random parts on the right hand side of (2.1) is merely a phenomenological simplification — microscopically, the two forces have the same origin (collision with the fluid constituents).

Assuming equipartition of energy, $\langle m\,\dot{x}^2\rangle = T$, with T the temperature,[6] one can derive the following time evolution for the displacement squared [2]:

$$\langle s(t)^2\rangle \equiv \langle [x(t)-x(0)]^2\rangle = \frac{2\,D}{\gamma_0}\left(\gamma_0\, t - 1 + e^{-\gamma_0\, t}\right) \approx \begin{cases} \dfrac{T}{m}\,t^2 & \left(t \ll \dfrac{1}{\gamma_0}\right) \\ 2\,D\,t & \left(t \gg \dfrac{1}{\gamma_0}\right) \end{cases} \tag{2.3}$$

where the diffusion constant D is related to the friction coefficient γ_0 by the Sutherland-Einstein relation,

$$D = \frac{T}{\gamma_0\, m}. \tag{2.4}$$

We can see that in the ballistic regime, $t \ll 1/\gamma_0$, the particle moves inertially ($s \sim t$) with the velocity determined by equipartition, $\dot{x} \sim \sqrt{T/m}$, while in the diffusive regime, $t \gg 1/\gamma_0$, the particle undergoes a random walk ($s \sim \sqrt{t}$). This is because the Brownian particle must be hit by a certain number of fluid particles to get substantially diverted from the direction of its initial velocity. The crossover time between the two regimes is the relaxation time

$$t_{\rm relax} \sim \frac{1}{\gamma_0}, \tag{2.5}$$

which characterizes the time scale for the Brownian particle to forget its initial velocity and thermalize. One can also derive the relation between the friction coefficient γ_0 and the size of the random force κ_0

$$\gamma_0 = \frac{\kappa_0}{2\,m\,T}, \tag{2.6}$$

which is the simplest example of the fluctuation-dissipation theorem and arises due to the fact that the frictional and random forces are of the same origin.

In n spatial dimensions, p and R in (2.1) are generalized to n component vectors and (2.2) is generalized to

$$\langle R_i(t)\rangle = 0, \qquad \langle R_i(t)R_j(t')\rangle = \kappa_0\, \delta_{ij}\, \delta(t-t'), \tag{2.7}$$

[6] We shall work in units where the Boltzmann constant $k_B = 1$.

where $i, j = 1, \ldots, n$. In the diffusive regime, the displacement squared goes as $\langle s(t)^2 \rangle \approx 2nDt$. The Sutherland-Einstein relation (2.4) and the fluctuation-dissipation relation (2.6) are independent of n.

Now let us go back to the case with one spatial dimension ($n = 1$). The Langevin equation (2.1), (2.2) captures certain essential features of physics, but nevertheless is too simple, for two reasons. It assumes that the friction is instantaneous and that there is no correlation between random forces at different times (eq. (2.2)). If the Brownian particle is not infinitely more massive than the fluid particles, these assumptions are no longer valid; friction will depend on the past history of the particle, and random forces at different times will not be fully independent. We can incorporate these effects by generalizing the simplest Langevin equation (2.1) to the so-called generalized Langevin equation [32, 33],

$$\dot{p}(t) = -\int_{-\infty}^{t} dt' \, \gamma(t - t') \, p(t') + R(t) + K(t). \tag{2.8}$$

Now the friction term depends on the past trajectory via the memory kernel $\gamma(t)$. The random force is taken to satisfy

$$\langle R(t) \rangle = 0, \qquad \langle R(t) R(t') \rangle = \kappa(t - t'), \tag{2.9}$$

where $\kappa(t)$ is some function. We have also now introduced an external force $K(t)$ that can be applied to the system.

To analyze the physical content of the generalized Langevin equation we Fourier transform (2.8) to obtain

$$p(\omega) = \frac{R(\omega) + K(\omega)}{\gamma[\omega] - i\omega}, \tag{2.10}$$

where $p(\omega), R(\omega), K(\omega)$ are Fourier transforms, *e.g.*,

$$p(\omega) = \int_{-\infty}^{\infty} dt \, p(t) \, e^{i\omega t}, \tag{2.11}$$

while $\gamma[\omega]$ is the Fourier-Laplace transform:

$$\gamma[\omega] = \int_{0}^{\infty} dt \, \gamma(t) \, e^{i\omega t}. \tag{2.12}$$

If we take the statistical average of (2.10), the random force vanishes because of the first equation in (2.9), and we obtain

$$\langle p(\omega) \rangle = \mu(\omega) K(\omega), \qquad \mu(\omega) \equiv \frac{1}{\gamma[\omega] - i\omega}. \tag{2.13}$$

$\mu(\omega)$ is called the admittance. So, we can determine the admittance $\mu(\omega)$, and thereby $\gamma[\omega]$, by measuring the response $\langle p(\omega) \rangle$ to an external force. In particular, if the external force is

$$K(t) = K_0 \, e^{-i\omega t}, \tag{2.14}$$

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then $\langle p(t)\rangle$ is simply

$$\langle p(t)\rangle = \mu(\omega)\, K_0\, e^{-i\omega t}. \tag{2.15}$$

For a quantity $\mathcal{O}(t)$, define the power spectrum $I_{\mathcal{O}}(\omega)$ by

$$I_{\mathcal{O}}(\omega) = \int_{-\infty}^{\infty} dt\, \langle \mathcal{O}(t_0)\, \mathcal{O}(t_0+t)\rangle\, e^{i\omega t}. \tag{2.16}$$

Note that $\langle \mathcal{O}(t_0)\mathcal{O}(t_0+t)\rangle$ is independent of t_0 in a stationary system. The knowledge of power spectrum is the same as that of 2-point function, because of the Wiener-Khintchine theorem

$$\langle \mathcal{O}(\omega)\mathcal{O}(\omega')\rangle = 2\pi\delta(\omega+\omega')I_{\mathcal{O}}(\omega). \tag{2.17}$$

Now consider the case without an external force, i.e., $K=0$. In this case, from (2.10),

$$p(\omega) = \frac{R(\omega)}{\gamma[\omega]-i\omega}. \tag{2.18}$$

Therefore, the power spectrum of p and that for R are related as

$$I_p(\omega) = \frac{I_R(\omega)}{|\gamma[\omega]-i\omega|^2}. \tag{2.19}$$

Combining (2.15) and (2.19), one can determine both $\gamma(t)$ and $\kappa(t)$ appearing in the Langevin equation (2.8) and (2.9) separately. However, as we will discuss in section 4, these two quantities are not independent but are related to each other by the fluctuation-dissipation theorem, which is the generalization of the relation (2.6).

For the generalized Langevin equation, what corresponds to the relaxation time (2.5) is

$$t_{\text{relax}} = \left[\int_0^\infty dt\, \gamma(t)\right]^{-1} = \frac{1}{\gamma[\omega=0]} = \mu(\omega=0). \tag{2.20}$$

If $\gamma(t)$ is sharply peaked around $t=0$, we can ignore the retarded effect of the friction term in (2.8) and write

$$\int_0^\infty dt'\, \gamma(t-t')\, p(t') \approx \int_0^\infty dt'\, \gamma(t')\cdot p(t) = \frac{1}{t_{\text{relax}}}\, p(t). \tag{2.21}$$

Then the Langevin equation reduces to the simple Langevin equation (2.1) and it is clear that t_{relax} corresponds to the thermalization time for the Brownian particle.

Another physically relevant time scale, the microscopic (or collision duration) time t_{coll}, is defined to be the width of the random force correlator function $\kappa(t)$. Specifically, let us define

$$t_{\text{coll}} = \int_0^\infty dt\, \frac{\kappa(t)}{\kappa(0)}. \tag{2.22}$$

If $\kappa(t) = \kappa(0)e^{-t/t_{\rm coll}}$, the right hand side of this precisely gives $t_{\rm coll}$. This $t_{\rm coll}$ characterizes the time scale over which the random force is correlated, and thus can be thought of as the time elapsed in a single process of scattering. In many cases,

$$t_{\rm relax} \gg t_{\rm coll}. \tag{2.23}$$

Typical examples for which (2.23) holds are the case where the particle is scattered occasionally by dilute scatterers, and the case where a heavy particle is hit frequently by much smaller particles [32]. As we will see later, for the Brownian motion dual to AdS black holes, the field theories are strongly coupled CFTs and (2.23) does not necessarily hold.

There is also a third natural time scale $t_{\rm mfp}$ given by the typical time elapsed between two collisions. In the kinetic theory, this mean free path time is typically $t_{\rm coll} \ll t_{\rm mfp} \ll t_{\rm relax}$; however in the case of present interest, this separation no longer holds, as we will see.

2.2 Bulk counterpart of Brownian motion

The AdS/CFT correspondence states that string theory in AdS_d is dual to a CFT in $(d-1)$ dimensions. In particular, the (planar) Schwarzschild-AdS black hole with metric

$$ds_d^2 = \frac{r^2}{\ell^2}\left[-h(r)\,dt^2 + d\vec{X}_{d-2}^2\right] + \frac{\ell^2}{r^2 h(r)}\,dr^2, \qquad h(r) = 1 - \left(\frac{r_H}{r}\right)^{d-1} \tag{2.24}$$

is dual to a CFT at a temperature equal to the Hawking temperature of the black hole,

$$T = \frac{1}{\beta} = \frac{(d-1)\,r_H}{4\pi\,\ell^2}. \tag{2.25}$$

In the above, ℓ is the AdS radius, and t, $\vec{X}_{d-2} = (X^1, \ldots, X^{d-2}) \in \mathbb{R}^{d-2}$ are the boundary coordinates.

In this black hole geometry (2.24), let us consider a fundamental string suspended from the boundary at $r = \infty$, straight down along the r direction, into the horizon at $r = r_H$; see figure 1. In the boundary CFT, this corresponds to having a very heavy external charged particle. The $\vec{X}_{d-2}$ coordinates of the string at $r = \infty$ in the bulk give the boundary position of the external particle. As we discussed above, such an external particle at finite temperature T is expected to undergo Brownian motion. The dual statement must be that the black hole environment in the bulk excites the modes on the string and, as the result, the endpoint of the string at $r = \infty$ exhibits a Brownian motion which can be modeled by a Langevin equation.

We study this motion of a string in the probe approximation where we ignore its backreaction on the background geometry. We assume that there is no B-field in the background, which is the case for AdS_3 based on D1- and D5-branes, and AdS_5 based on D3-branes. If we take the string coupling g_s to be very small, the interaction of the string with the thermal gas of closed strings in the bulk of the AdS space can be ignored; the only possible region with appreciable interaction is near the black hole horizon which the string is dipping into.

Let us slightly generalize (2.24) for a little while and consider the following metric:

$$ds^2 = g_{\mu\nu}(x)\,dx^\mu\,dx^\nu + G_{IJ}(x)\,dX^I\,dX^J, \tag{2.26}$$

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where $x^\mu = t, r$ and $I, J = 1, \dots, d-2$. For the spacetimes of interest, both $g_{\mu\nu}$ and G_{IJ} are independent of X^I.[7] Now, we stretch a string along the r direction and consider small fluctuation of it in the transverse directions X^I. The action for the string is simply the Nambu-Goto action in the absence of B-field. In the gauge where the world-sheet coordinates are identified with the spacetime coordinates $x^\mu = t, r$, the transverse fluctuations X^I become functions of x^μ: $X^I = X^I(x)$. If we expand the Nambu-Goto action up to quadratic order in X^I, we obtain

$$\begin{aligned} S_{\rm NG} &= -\frac{1}{2\pi\alpha'}\int d^2x \sqrt{-\det \gamma_{\mu\nu}} \\ &\approx -\frac{1}{4\pi\alpha'}\int d^2x \sqrt{-g(x)}\, g^{\mu\nu}(x)\, G_{IJ}(x)\, \frac{\partial X^I}{\partial x^\mu}\frac{\partial X^J}{\partial x^\nu} \equiv S_{\rm NG}^{(2)}, \end{aligned} \tag{2.27}$$

where $\gamma_{\mu\nu}$ is the induced metric, $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, and $g = \det g_{\mu\nu}$. In the last line we dropped the constant term that does not depend on X^I. The quadratic approximation is of course valid as long as the scalars X^I do not fluctuate too far from their equilibrium value (taken here to be $X^I = 0$).[8] In fact, this quadratic fluctuation Lagrangian for the world-sheet scalars (2.27) can be thought of as taking the non-relativistic limit; the Nambu-Goto action is after all a non-polynomial action in the velocities $\partial_t X^I$ and we are expanding in the regime $|\partial_t X^I| \ll 1$. Therefore, we expect (and will see) that the dual Langevin dynamics on the boundary will also be a non-relativistic one, which is precisely what we reviewed in the previous subsection. For most of the paper, we will use this quadratic action $S_{\rm NG}^{(2)}$ to study the fluctuations of the string.[9] The equation of motion derived from (2.27) is

$$0 = \nabla^\mu(G_{IJ}\,\partial_\mu X^I) = \frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\, G_{IJ}\,\partial_\nu X^J), \tag{2.28}$$

where ∇_μ is the covariant derivative with respect to $g_{\mu\nu}$. Note that this is not the same as the Klein-Gordon equation in the spacetime (2.26), which would involve not just $\partial_\mu = \partial/\partial x^\mu$ but also $\partial_I = \partial/\partial X^I$.

Returning to the AdS black hole metric (2.24), we focus first on the AdS$_3$ ($d=3$) case for simplicity (we will discuss AdS$_d$ with general d in section 5) and study the motion of a string in the black hole background. In this case, the metric (2.24) becomes the nonrotating BTZ black hole:

$$ds^2 = -\frac{r^2 - r_H^2}{\ell^2}\, dt^2 + \frac{r^2}{\ell^2}\, dX^2 + \frac{\ell^2}{r^2 - r_H^2}\, dr^2. \tag{2.29}$$

[7] We will mainly focus on planar black holes in AdS corresponding to thermal field theories on $\mathbb{R}^{d-1,1}$ when the transverse directions to the string X^I are indeed Killing directions in the bulk.

[8] One can show that when the modes on the string are thermally excited in a black hole background at temperature T, this quadratic approximation is valid outside the black hole except for the region within $\sqrt{\alpha'}$ away from the horizon.

[9] In appendix D, we will consider the next leading terms (quartic terms) when we estimate the mean free path time $t_{\rm mfp}$.

For the usual BTZ black hole, X is written as $X = \ell\phi$ where $\phi \cong \phi + 2\pi$, but here we are taking $X \in \mathbb{R}$. The Hawking temperature (2.25) is, in this case,

$$T \equiv \frac{1}{\beta} = \frac{r_H}{2\pi\,\ell^2}. \tag{2.30}$$

In terms of the tortoise coordinate r_*, the metric (2.29) becomes

$$ds^2 = \frac{r^2 - r_H^2}{\ell^2}(-dt^2 + dr_*^2) + \frac{r^2}{\ell^2}\,dX^2, \qquad r_* \equiv \frac{\ell^2}{2r_H}\ln\left(\frac{r - r_H}{r + r_H}\right). \tag{2.31}$$

For the BTZ metric (2.29), the equation of motion (2.28) becomes

$$\left[-\partial_t^2 + \frac{r^2 - r_H^2}{\ell^4\,r^2}\,\partial_r\Big(r^2\,(r^2 - r_H^2)\,\partial_r\Big)\right] X(t,r) = 0. \tag{2.32}$$

As usual, we proceed by expanding X in modes. Let us set

$$X(t,r) = e^{-i\omega t} f_\omega(r). \tag{2.33}$$

Then the equation of motion (2.32) can be written as

$$\left[\nu^2 + \frac{\rho^2 - 1}{\rho^2}\,\partial_\rho\Big(\rho^2(\rho^2 - 1)\partial_\rho\Big)\right] f_\omega = 0, \tag{2.34}$$

where we defined dimensionless quantities

$$\rho \equiv \frac{r}{r_H}, \qquad \nu \equiv \frac{\ell^2\omega}{r_H} = \frac{\beta\omega}{2\pi}. \tag{2.35}$$

One can see that the linearly independent solutions to (2.34) are given by

$$f_\omega^{(\pm)} = \frac{1}{1 \pm i\nu}\,\frac{\rho \pm i\nu}{\rho}\left(\frac{\rho - 1}{\rho + 1}\right)^{\pm i\nu/2} = \frac{1}{1 \pm i\nu}\,\frac{\rho \pm i\nu}{\rho}e^{\pm i\omega r_*}. \tag{2.36}$$

The normalization in (2.36) was chosen so that, near the horizon,

$$f_\omega^{(\pm)} \sim e^{\pm i\omega r_*} \qquad (\rho \sim 1), \tag{2.37}$$

and hence the solutions are written naturally in terms of ingoing ("+" sign) and outgoing ("−" sign) modes.

2.3 Boundary conditions and cut-offs

Before proceeding with the analysis of the fluctuations of the scalar field X in the BTZ geometry, it is useful to understand the boundary conditions we want to impose on the fields. While we are actually interested in the world-sheet theory of the probe string, it is clear that we can use the usual AdS/CFT rules to understand the boundary conditions; in the static gauge the induced metric on the string world-sheet inherits the geometric characteristics of an asymptotically AdS_2 spacetime.

Usually in Lorentzian AdS/CFT one chooses to use normalizable boundary conditions [34] for the modes. However, in the present case, that would correspond to a string extending all the way to $\rho = \infty$, which would mean that the mass of the external particle is infinite and there would be no Brownian motion. So, instead, we have to impose a UV cut-off[10] near the boundary to make the mass finite. Specifically, we implement this by means of a Neumann boundary condition $\partial_r X = 0$ at the cut-off surface[11]

$$\rho = \rho_c \gg 1, \qquad \text{or} \qquad r = r_c \equiv r_H \rho_c. \tag{2.38}$$

The relation between the UV cut-off $\rho = \rho_c$ and the mass m of the external particle is easily computed from the tension of the string:

$$m = \frac{1}{2\pi\alpha'} \int_{r_H}^{r_c} dr \sqrt{g_{tt}\, g_{rr}} = \frac{r_c - r_H}{2\pi\alpha'} = \frac{\ell^2 (\rho_c - 1)}{\alpha' \beta} \approx \frac{\ell^2 \rho_c}{\alpha' \beta}. \tag{2.39}$$

Setting

$$f_\omega(\rho) = A \Big[f_\omega^{(+)}(\rho) + B f_\omega^{(-)}(\rho) \Big], \tag{2.40}$$

with constants A and B, we obtain, on implementing the Neumann boundary condition $\partial_\rho f_\omega|_{\rho=\rho_c} = 0$,

$$B = \frac{1 - i\nu}{1 + i\nu} \frac{1 + i\rho_c \nu}{1 - i\rho_c \nu} \left(\frac{\rho_c - 1}{\rho_c + 1} \right)^{i\nu} \equiv e^{i\theta_\omega}. \tag{2.41}$$

Note that this is a pure phase. This in particular means that, in the near-horizon region $r \sim r_H$, we have, because of (2.37),

$$X(t, r) = f_\omega \, e^{-i\omega t} \sim e^{-i\omega(t - r_*)} + e^{i\theta_\omega} \, e^{-i\omega(t + r_*)}. \tag{2.42}$$

The first term is a mode which is outgoing at the horizon, while the second term is a mode reflected at $\rho = \rho_c$ and falling back into the horizon, with phase shift $e^{i\theta_\omega}$. The fact that the outgoing and ingoing modes have the same amplitude means that the AdS black hole, which Hawking radiates, can be in thermal equilibrium at temperature T [35].

To regulate the theory, we need to introduce another cut-off near the horizon $\rho = 1$. Specifically, we cut off the geometry by putting an IR cut-off ("stretched horizon") at ρ_s

$$\rho_s = 1 + 2\,\epsilon, \qquad \epsilon \ll 1. \tag{2.43}$$

If we impose a Neumann boundary condition[12] at ρ_s, we have, just as (2.41),

$$B = \frac{1 - i\nu}{1 + i\nu} \frac{1 + i(1 + 2\epsilon)\nu}{1 - i(1 + 2\epsilon)\nu} \, \epsilon^{i\nu} \approx \epsilon^{i\nu} = e^{-i\nu \log(1/\epsilon)}. \tag{2.44}$$

[10] We use the terms "UV" and "IR" with respect to the boundary energy. In this terminology, in the bulk, UV means near the boundary and IR means near the horizon.

[11] In the AdS/QCD context, one can think of the cut-off being determined by the location of the flavour brane, whose purpose again is to introduce dynamical (and therefore finite mass) quarks into the field theory.

[12] One could also take a Dirichlet boundary condition, but in the $\epsilon \to 0$ limit this would not make a difference.

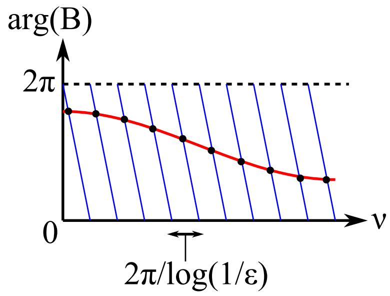

Figure 2. Boundary conditions at infinity and horizon. First, the UV boundary condition (2.41) fixes $\arg B$ to lie, say, on the red line; at this point the possible values of ν are continuous. Further imposing the IR boundary condition (2.44) makes the possible values of ν discrete (black dots).

If we require (2.41) only, then we determine B as a function of ν, but at this point ν can take any value and is continuous. If $\epsilon \ll 1$, further requiring (2.44) effectively makes the possible values of ν discrete, and the discreteness is given by $\Delta\nu = 2\pi/\log(1/\epsilon) \ll 1$; see figure 2. In terms of the frequency ω, the discreteness is

$$\Delta\omega = \frac{4\pi^2}{\beta\log(1/\epsilon)}. \tag{2.45}$$

In other words, we have the following density of states:

$$\mathcal{D}(\omega) = \frac{1}{\Delta\omega} = \frac{\beta\log(1/\epsilon)}{4\pi^2}. \tag{2.46}$$

All we have achieved by putting the regulator near the horizon is to discretize the continuum spectrum which naturally occurs when considering horizon dynamics.

Having done regularization, we can find a normalized basis of modes and start quantizing $X(t,r)$ by expanding it in those modes. This is standard as in the case of a scalar field obeying the Klein-Gordon equation; for details we refer the reader to appendix A. The upshot of the calculation is

$$X(t,r) = \sum_{\omega>0}\left[a_\omega u_\omega(t,\rho) + a_\omega^\dagger u_\omega(t,\rho)^*\right], \tag{2.47}$$

where the summation is over ω discretized according to (2.45). The normalized basis u_ω is

$$u_\omega(t,\rho) = \sqrt{\frac{\alpha'\beta}{2\,\ell^2\,\omega\,\log(1/\epsilon)}}\left[f_\omega^{(+)}(\rho) + B\,f_\omega^{(-)}(\rho)\right]e^{-i\omega t}, \tag{2.48}$$

where B is given by (2.41). The expansion coefficients a_ω satisfy the commutation relations

$$[a_\omega, a_{\omega'}] = [a_\omega^\dagger, a_{\omega'}^\dagger] = 0, \qquad [a_\omega, a_{\omega'}^\dagger] = \delta_{\omega\omega'}. \tag{2.49}$$

2.4 The boundary-bulk dictionary

Given the behavior of quantum modes on the probe string in the bulk, we can work out the dynamics of the endpoint, which corresponds to a test quark in the thermal CFT plasma. To understand the precise dictionary we look at the wave-functions of the world-sheet fields ($X(t,\rho)$ in the BTZ geometry) in the two interesting regions: (i) near the black hole horizon and (ii) close to the boundary.

From (2.37), near the horizon ($\rho \sim 1$), the expansion (2.47) becomes

$$X(t,\rho\sim 1) \approx \sum_{\omega>0} \sqrt{\frac{\alpha'\,\beta}{2\,\ell^2\,\omega\,\log(1/\epsilon)}}\left[\left(e^{-i\omega(t-r_*)} + e^{i\theta_\omega}\,e^{-i\omega(t+r_*)}\right)\,a_\omega + \text{h.c.}\right]. \tag{2.50}$$

We see that the operators a_ω are directly related to the amplitude for the outgoing modes, $e^{-i\omega(t-r_*)}$, near the horizon. On the other hand, at the UV cut-off $\rho = \rho_c$, which we have chosen to be the location of the regularized boundary, (2.47) becomes

$$x(t) \equiv X(t,\rho_c) = \sum_{\omega>0} \sqrt{\frac{2\alpha'\,\beta}{\ell^2\,\omega\,\log(1/\epsilon)}}\left[\frac{1-i\nu}{1-i\rho_c\nu}\left(\frac{\rho_c-1}{\rho_c+1}\right)^{i\nu/2}\,e^{-i\omega t}\,a_\omega + \text{h.c.}\right] \tag{2.51}$$

This we will interpret as the position of the external particle (test quark) in the boundary theory. Here, the operators a_ω are related to the Fourier coefficients of $x(t)$.

Using the above relation between (2.50) and (2.51), one can predict the correlators for the outgoing modes near the horizon, $\langle a_{\omega_1} a^\dagger_{\omega_2} \dots\rangle$ *etc.*, from the boundary correlators $\langle x(t_1)\, x(t_2)\dots\rangle$ in field theory. In particular, if we would be able to make a very precise measurement of Brownian motion in field theory, we could *in principle* predict the precise state of the radiation that comes out of the black hole. In this way, we can learn about the physics of quantum black holes in the bulk from the boundary data. This of course requires us to compute the correlation function for the test particle's position in a strongly coupled medium, which is a difficult task that we will not undertake here.

However, at the semiclassical level, we can utilize this dictionary to rather go from the bulk to the boundary and learn about the boundary Brownian motion from the bulk data. This is possible because, semiclassically, the state of the outgoing modes near the horizon is given by the usual Hawking radiation. As argued in [36, 37], the modes on the string world-sheet which impinges on the black hole horizon are thermally excited with a black-body spectrum determined by the Hawking temperature. The quickest way to see this is to note that one can view our analysis of the fluctuations of the string world-sheet (2.27) as studying the dynamics of massless, free scalars in a two dimensional black hole background.[13] Standard quantization of quantum fields in curved spacetime [38] will lead to the modes of the fields X^I being thermally excited at the Hawking temperature of this induced world-sheet geometry which is the same as that for the BTZ black hole. In particular, it follows that the outgoing mode correlators are determined by the thermal

[13]The induced metric on the string world-sheet clearly has a horizon and is an asymptotically AdS_2 spacetime.

density matrix

$$\rho_0 = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}, \qquad H = \sum_{\omega>0} \omega\, a_\omega^\dagger a_\omega. \tag{2.52}$$

Note that, as we discussed above (2.26), here we are ignoring the interaction of the string with the thermal gas of closed strings in the bulk ($r_H < r < r_c$) of the black hole background. Namely, we regard the above density matrix (2.52) as solely due to the interaction with the horizon. However, even if we took into account the weak interaction with the thermal gas of closed strings, the density matrix would still be given with very good accuracy by (2.52) because, at each value of r, the thermal gas is in thermal equilibrium at the local Hawking temperature and so is the string.

As long as we stay in the semi-classical approximation we can use the observations mentioned above to go from the bulk to the boundary and derive the Brownian motion of the external particle in the field theory. That is, instead of using the boundary field theory to compute the correlation function of the quantum operators a and $a^\dagger$, we can use the fact that these correlators are determined by the thermal physics of black holes and utilize them to compute the boundary correlation functions. In particular, we propose to use the knowledge (2.52) about the outgoing mode correlators in the bulk, to predict the nature of Brownian motion that the external particle on the boundary undergoes. Thus by using the standard physics of black holes we will be able to determine the functions $\gamma(t)$, $\kappa(t)$ appearing in the Langevin equation (2.8).

3 Hawking radiation and Brownian motion

In the previous section, we used the AdS/CFT correspondence to set-up a dictionary translating the information about the boundary Brownian particle into corresponding data regarding the outgoing modes (the world-sheet oscillators a_I and $a_I^\dagger$ of the fluctuations X^I) in the bulk. In this section, we explicitly derive the correlation function for the position of the test particle. We *assume* that the outgoing modes are the usual Hawking radiation with the density matrix (2.52) and derive the result that the endpoint at $\rho = \rho_c \gg 1$ indeed undergoes a Brownian motion.

3.1 Brownian motion of the boundary endpoint

Let us now consider the motion of the endpoint of the string at $\rho = \rho_c \gg 1$. We will determine the behavior by computing its displacement squared, which corresponds to (2.3). In the canonical ensemble specified by the density matrix (2.52), the relevant expectation values are given by the Bose-Einstein distribution:

$$\langle a_\omega^\dagger\, a_{\omega'} \rangle = \mathrm{Tr}\left(\rho_0\, a_\omega^\dagger\, a_{\omega'}\right) = \frac{\delta_{\omega\,\omega'}}{e^{\beta\omega} - 1}. \tag{3.1}$$

Using this and (2.51), we compute

$$\langle x(t)\,x(0)\rangle = \langle X(t,\rho_c)\,X(0,\rho_c)\rangle = \frac{2\alpha'\,\beta}{\ell^2\,\log(1/\epsilon)}\sum_{\omega>0}\frac{1}{\omega}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}\left(\frac{2\cos\omega t}{e^{\beta\omega}-1}+e^{-i\omega t}\right)$$
$$= \frac{\alpha'\,\beta^2}{2\pi^2\,\ell^2}\int_0^\infty\frac{d\omega}{\omega}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}\left(\frac{2\cos\omega t}{e^{\beta\omega}-1}+e^{-i\omega t}\right). \tag{3.2}$$

We are using the rescaled frequency ν defined in (2.35) throughout to avoid clutter. In going to the second line, we utilized the density of states determined in (2.45) to rewrite the sum as an integral. From this, we compute the displacement of the endpoint as:

$$s^2(t) \equiv \langle[x(t)-x(0)]^2\rangle = \frac{2\alpha'\beta^2}{\pi^2\ell^2}\int_0^\infty\frac{d\omega}{\omega}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}\coth\frac{\beta\omega}{2}\,\sin^2\frac{\omega t}{2}. \tag{3.3}$$

This has a logarithmic UV divergence. Because this divergence is coming from the zero point energy (the $e^{-i\omega t}$ term in (3.2)), which exists even at zero temperature, we simply regularize it by normal ordering the $a, a^\dagger$ oscillators $(:a_\omega a_\omega^\dagger: \equiv :a_\omega^\dagger a_\omega:)$. When so regularized, the correlator (3.2) becomes

$$\langle :x(t)x(0):\rangle = \frac{\alpha'\beta^2}{\pi^2\,\ell^2}\int_0^\infty\frac{d\omega}{\omega}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}\,\frac{\cos\omega t}{e^{\beta\omega}-1}, \tag{3.4}$$

and the displacement squared (3.3) becomes[14,15]

$$s^2_{\rm reg}(t) \equiv \langle :[x(t)-x(0)]^2:\rangle = \frac{4\alpha'\,\beta^2}{\pi^2\,\ell^2}\int_0^\infty\frac{d\omega}{\omega}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}\,\frac{\sin^2\frac{\omega t}{2}}{e^{\beta\omega}-1}. \tag{3.5}$$

We analytically evaluate this integral in appendix B. For the present purposes we will only record the result for $\rho_c \gg 1$, which is all that is relevant for the physics of the boundary field theory. We find the following behavior:

$$s^2_{\rm reg}(t) \approx \begin{cases} \dfrac{\alpha'}{\ell^2\,\rho_c}\,t^2 \approx \dfrac{T}{m}\,t^2 & (t \ll t_c), \\[2ex] \dfrac{\alpha'}{\pi\,\ell^2\,T}\,t & (t \gg t_c). \end{cases} \tag{3.6}$$

[14] Note that regularization by normal-ordering does not preserve the KMS relations except in the classical limit.

[15] Another way to regularize the correlator is to use the canonical correlator introduced in (4.7). Using (4.12), one can derive

$$s_c^2(t) \equiv \langle [x(t)-x(0)];[x(t)-x(0)]\rangle = \frac{2\alpha'\beta^2}{\pi^2\,\ell^2}\int_{-\infty}^\infty\frac{d\omega}{\omega^2}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}\sin^2\frac{\omega t}{2}$$
$$= \frac{\alpha'\beta^2}{\pi\ell^2}\left[(|t|/\beta)-(1-\rho_c^{-2})(1-e^{-2\pi|t|/\beta\rho_c})\right].$$

This is finite and has exactly the same short- and long-time behaviors as in (3.6). Note, however, that the divergence (3.3) is related to the well-known fact that the fluctuation of the position of a string always diverges [39].

So, we observe two regimes, the ballistic and diffusive regimes, exactly as for the standard Brownian motion (2.3). The crossover time t_c is given by

$$t_c \sim \beta\, \rho_c \sim \frac{\alpha'\, m}{\ell^2\, T^2}. \tag{3.7}$$

In the ballistic regime, $t \ll t_c$, the coefficient of t^2 in (3.6) is exactly the same as (2.3) determined by the equipartition of energy $\dot{x} \sim \sqrt{T/m}$. In fact, one can say much more; in appendix C, we show that, if $\rho_c \gg 1$, the probability distribution $f(p)$ for the "momentum" $p \equiv m\,\dot{x}$ of the endpoint is exactly equal to the Maxwell-Boltzmann distribution for non-relativistic particles,

$$f(p) \propto e^{-\beta E_p}, \qquad E_p = \frac{p^2}{2m}. \tag{3.8}$$

In the diffusive regime, $t \gg t_c$, we find a diffusion constant (one half of the coefficient of t)

$$D_{\mathrm{AdS}_3} = \frac{\alpha'}{2\pi\ell^2 T}\ . \tag{3.9}$$

A priori this looks counterintuitive because it is inversely proportional to temperature T and implies that the random walk becomes more vigorous at lower temperature. However, this is consistent with the known results for test quarks moving in the thermal $\mathcal{N}=4$ super Yang-Mills plasma [20–22, 24]. For example, refs. [20, 22] considered a heavy particle on the boundary moving at a constant speed v under the influence of an external force. One can compute the friction acting on it from the bulk using the Nambu-Goto action, where a string is moving at velocity v, trailing along the boundary. It is easy to generalize their computation to AdS_d with general d, the result being[16]

$$\dot{p} = -\frac{8\pi\,\ell^2\,T^2}{(d-1)^2\,\alpha'}\,\frac{v}{(1-v^2)^{2/(d-1)}}, \qquad p = \frac{m\,v}{\sqrt{1-v^2}}. \tag{3.10}$$

In the non-relativistic limit, $v \ll 1$, this means that the friction constant is

$$\gamma_0^{\mathrm{AdS}_d} = \frac{8\pi\,\ell^2\,T^2}{(d-1)^2\,\alpha'\,m}. \tag{3.11}$$

If we use the Sutherland-Einstein relation (2.6),[17] we obtain the diffusion constant

$$D_{\mathrm{AdS}_d} = \frac{(d-1)^2\,\alpha'}{8\pi\,\ell^2\,T}, \tag{3.12}$$

which agrees with (3.9) for $d=3$.

One can give an intuitive explanation for the reason why the diffusion constant is inversely proportional to T from the boundary viewpoint of Brownian motion. The random

[16] Although this is a straightforward generalization of [20, 22] and the general formalism has been laid out in [23], it seems to us that this result has not appeared explicitly in the literature.

[17] Note that, as explained around (2.7), the relation (2.6) does not depend on d.

walk behavior of Brownian motion is due to frequent collisions of the Brownian particle with the fluid particles. In particular, after n steps (collisions), the distance s that a random-walk particle covers scales as $\sqrt{n}\,L_{\rm mfp}$, where the mean free path $L_{\rm mfp}$ is the typical length traveled between the collisions, i.e., it provides a scale for the system. For the thermal system under consideration we have, $L_{\rm mfp} \sim 1/T$, because this is the only scale available in a CFT at temperature T.[18] After n collisions, the time elapsed is given by $t \sim n/T$, since the time between collisions is also given by $L_{\rm mfp} \sim 1/T$. So, putting things together, we have $s \sim \sqrt{tT} \cdot 1/T = \sqrt{t/T}$, namely, $s^2 \sim t/T$ which is exactly what we infer from (3.12).

From the bulk point of view, on the other hand, one can give a physical explanation for $D \sim 1/T$ as follows. Near the horizon ($\rho \sim 1$), the Nambu-Goto action (2.27) becomes

$$S_{\rm NG} \approx \frac{\pi\,\ell^2\,T^2}{\alpha'} \int dt\, dr_*\left[(\partial_t X)^2 - (\partial_{r_*} X)^2\right]. \tag{3.13}$$

This is the same as the action for a string in flat space, with α' replaced by $\alpha'_{\rm eff} = \alpha'/(4\pi^2\ell^2T^2)$. This means that the size of the fluctuations in X^2 is proportional to $\alpha'_{\rm eff} \sim T^{-2}$. The Boltzmann factor of Hawking radiation gives an additional factor of $1/(e^{\beta\,\omega} - 1)$ which scales as T at low frequency. Altogether, near the horizon, the fluctuations scale with temperature as $X^2 \sim T^{-1}$. When a fluctuation propagates to $\rho = \rho_c$, a greybody factor damps the fluctuation. However, as one can see from (2.51), the damping is $\mathcal{O}(1)$ for very small frequency. This leads to $x^2 = X(\rho = \rho_c)^2$ being $\sim T^{-1}$. The reason why very low frequency modes can reach $\rho = \rho_c$ undamped is that X is an isometry direction and very low frequency X modes can propagate at almost no cost in energy.

A natural question to ask is what happens to this T^{-1} scaling as $T \to 0$, as we expect that the endpoint should not fluctuate at $T = 0$. This can be understood by realizing that for a mode to propagate to $\rho = \rho_c$ undamped, it should be the lightest mode in the problem. In particular, only modes whose frequencies are lower than the thermal scale (which goes to zero as $T \to 0$) can propagate without damping. Translating to real time dynamics, this means that one needs to wait until $t \sim t_c$ to see the diffusive regime; but since $t_c \propto T^{-2} \to \infty$ as $T \to 0$ we never enter that regime and the motion is always ballistic as expected.

Thus, we have demonstrated that the endpoint of the string at $\rho = \rho_c$ indeed behaves like a Brownian particle; it shows ballistic and diffusive regimes, just as for the usual Brownian motion. We would now like to understand the Langevin equation from the bulk perspective. As we will see below, the Langevin equation governing this Brownian motion turns out to be not of the simplest type (2.1) and (2.2), but rather the generalized one (2.8) indicating that the precise nature of the random kick encountered by the Brownian particle depends on the past history of its trajectory.

[18] The precise value of the $L_{\rm mfp}$ depends on the strength of the field theory coupling, but the temperature dependence follows via dimensional analysis. In fact, in appendix D, we estimate the mean free path time to be $t_{\rm mfp} \sim 1/(\sqrt{\lambda}\,T)$, where $\lambda \sim \ell^4/\alpha'^2$. If the plasma constituents are moving at the speed of light, this means that $L_{\rm mfp} \sim 1/(\sqrt{\lambda}\,T)$. With this value of $L_{\rm mfp}$, we can even recover the λ dependence of (3.12).

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3.2 Forced motion and the holographic Langevin equation

As we discussed in subsection 2.1, the generalized Langevin equation (2.8) has two functional parameters: the memory kernel $\gamma(t)$ and the auto-correlation function $\kappa(t)$, related to the dissipative and stochastic components, respectively. We would like to determine these functions from the holographic viewpoint for the probe string in the black hole background. In order to do so, we will first determine $\gamma(t)$, or equivalently $\mu(\omega)$. Once we know $\mu(\omega)$, we can compute $\kappa(\omega)$ by using equation (2.19) and the $\langle x\, x\rangle$ correlator (3.2) (or (3.4)).

We first turn to the determination of $\mu(\omega)$. Consider applying an external force on the Brownian particle as in (2.14); from the response to this force we can read off $\mu(\omega)$ using (2.15). So the natural question is what external force is to be applied to the string endpoint. As in the AdS/QCD set-ups, we can realize such forced motion by placing a "flavor" D-brane at the UV cut-off $\rho = \rho_c$ and by turning on world-volume electric field on it. Since the endpoint of the string is charged, this will exert the desired force on the Brownian particle.

So, let us consider the Nambu-Goto action (2.27) in the general metric (2.26), and add to it the following boundary term

$$S_{\rm bdy} = \oint A(x,X), \tag{3.14}$$

which corresponds to turning on world-volume field on the flavor D-brane (which is placed at the UV cut-off ρ_c). Here, $A(x,X)$ is a 1-form defined on the flavor D-brane world-volume. We again work in the gauge where the world-sheet coordinates are identified with the spacetime coordinates $x^\mu = t, r$. We have $X^I = X^I(x)$ and

$$S_{\rm bdy} = \oint [A_t(x,X) + A_I(x,X)\,\dot{X}^I]\,dt, \tag{3.15}$$

where t is taken to be the coordinate along the boundary as before (or equivalently, the boundary is at $r = $ const.). The equation of motion one obtains for the total action $S_{\rm NG}^{(2)} + S_{\rm bdy}$ at the boundary is

$$\sqrt{-\widetilde{g}}\; n^\mu\, G_{IJ}\,\partial_\mu X^J - 2\pi\alpha' \left(F_{It} + F_{IJ}\,\partial_t X^J\right) = 0, \tag{3.16}$$

where $\widetilde{g}_{\mu\nu}$ is the induced metric on the boundary, n^μ is the outward-pointing unit normal to the boundary, and $F_{It} = \partial_I A_t - \partial_t A_I$, $F_{IJ} = \partial_I A_J - \partial_J A_I$.

Returning to the simple setting of the BTZ geometry (2.29), the equation of motion for the string in the presence of this additional gauge field is

$$\rho^2(\rho^2-1)\,\partial_\rho X = \frac{2\pi\alpha'\,\ell^4}{r_H^3}\,F_{Xt} \qquad\qquad \text{at } \rho = \rho_c. \tag{3.17}$$

For the world-volume field F_{Xt} we choose an oscillating electric field with frequency ω:

$$F_{Xt} \equiv E = E_0\, e^{-i\omega t}, \tag{3.18}$$

motivated by (2.14). We now want to compute how the string, in particular its endpoint $X(t, \rho_c) = x(t)$, moves under the influence of this external force in order to compute the admittance $\mu(\omega)$.

As before, the solution to the bulk equation of motion can be written as a linear combination of the modes $f_\omega^{(\pm)}(\rho)$:

$$X(t, \rho) = \left[A' f_\omega^{(+)}(\rho) + B' f_\omega^{(-)}(\rho)\right] e^{-i\omega t}. \tag{3.19}$$

To determine the coefficients A', B', we need to impose a boundary condition at the horizon, in addition to the boundary condition (3.17) at $\rho = \rho_c$. Effectively all that the external field has done was to modify the Neumann boundary condition, which we imposed earlier on the cut-off surface, to a mixed boundary condition.

In the semiclassical approximation, the boundary condition near the horizon is such that outgoing modes are always thermally excited because of Hawking radiation, while the ingoing modes can be arbitrary.[19] From (2.37), the coefficients A' and B' correspond to outgoing and ingoing modes respectively. Therefore, the boundary condition at the horizon is that A' is thermally excited. However, because the radiation is random, the phase of A' takes random values and, on average, A' vanishes: $\langle A' \rangle = 0$. Recall that it is such averaged quantities that we are interested in; the admittance $\mu(\omega)$ is obtained by suitably averaging over the ensemble, cf. (2.15).

Requiring the boundary condition (3.17), with electric field (3.18) at $\rho = \rho_c$ and the condition that $\langle A' \rangle = 0$, we determine the average values of A', B' to be

$$\langle A' \rangle = 0, \qquad \langle B' \rangle = \frac{2i\pi\alpha' \ell^4}{r_H^3} \frac{1 - i\nu}{\nu\,(1 - i\rho_c\nu)} \left(\frac{\rho_c - 1}{\rho_c + 1}\right)^{i\nu/2} E_0\ . \tag{3.20}$$

From this we infer that the average value of X at the UV cut-off $\rho = \rho_c$ is

$$\langle x(t) \rangle = \langle X(t, \rho_c) \rangle = \frac{2i\pi\alpha' \ell^4}{r_H^3} \frac{1 - i\nu/\rho_c}{\nu\,(1 - i\rho_c\nu)} E_0\, e^{-i\omega t}\,, \tag{3.21}$$

which in turn implies that the average value of the momentum $p = m\dot{x}$ is

$$\langle p(t) \rangle = \frac{2\pi\alpha' \ell^4\, m\omega}{r_H^3} \frac{1 - i\nu/\rho_c}{\nu\,(1 - i\rho_c\nu)} E_0\, e^{-i\omega t} = \frac{\alpha' \beta^2\, m}{2\pi\,\ell^2} \frac{1 - i\nu/\rho_c}{1 - i\rho_c\nu} E_0\, e^{-i\omega t}\ . \tag{3.22}$$

Comparing this with (2.15), we obtain the admittance

$$\mu(\omega) = \frac{1}{\gamma[\omega] - i\omega} = \frac{\alpha' \beta^2\, m}{2\pi\,\ell^2} \frac{1 - i\nu/\rho_c}{1 - i\rho_c\nu}. \tag{3.23}$$

A simple check on the consistency of these computations is to compute the energy flow along the string falling into the horizon, which must be equal to the work done by the external force. For the theory (2.27), the stress-energy tensor is

$$T^\mu_\nu = \frac{1}{2\pi\,\alpha'} \left(g^{\mu\kappa}\,\delta^\lambda_\nu - \frac{1}{2}\,\delta^\mu_\nu\, g^{\kappa\lambda}\right) G_{IJ}\,\partial_\kappa X^I \partial_\lambda X^J. \tag{3.24}$$

[19] If one ignores the thermal excitations of the outgoing modes and sets them to zero, this boundary condition becomes the so-called purely ingoing boundary condition.

Because we are working in static gauge this world-sheet stress-energy tensor measures the spacetime energy. In the case of the BTZ spacetime (2.29), the flow of energy along the r direction is

$$\sqrt{-g}\,T_t^r = \frac{r^2\,(r^2 - r_H^2)}{2\pi\,\alpha'\,\ell^4}\,\mathrm{Re}[\partial_r \overline{X}\,\partial_t X] = \frac{r_H^3\,\rho^2(\rho^2-1)}{2\pi\,\alpha'\,\ell^4}\,\mathrm{Re}[\partial_\rho \overline{X}\partial_t X]. \tag{3.25}$$

Here, we replaced $\partial_\kappa X \partial_\lambda X \to \mathrm{Re}[\partial_\kappa \overline{X} \partial_\lambda X]$ so as to work directly with complex fields. Consider the solution for $X(t,\rho)$ as in (3.19) with the coefficients A', B' given by the average value (3.20) (we ignore thermal fluctuations in replacing the amplitudes by their average). Then, (3.25) evaluates to[20]

$$\sqrt{-g}\,T_t^r = \frac{2\pi\,\alpha'\,\ell^2\,E_0^2}{r_H^2}\frac{1+\nu^2}{1+\rho_c^2\,\nu^2}\,. \tag{3.26}$$

On the other hand, the work done per unit time (namely, power) by the electric field E acting on the endpoint at $X(t,\rho_c)$ is

$$\mathrm{Re}[\overline{E}\,\partial_t X(t,\rho_c)]\,, \tag{3.27}$$

where E is given by (3.18). For $X(t,\rho)$ in (3.19), it is easy to check that this equals (3.26). Hence indeed as expected, the work done by the external force is transmitted down the string into the black hole horizon and the energy is thus dissipated away.

3.3 The holographic auto-correlation function and time scales

We now turn to the computation of the random force correlator, $\kappa(\omega)$. From the $\langle x\,x\rangle$ correlator (3.4), we can compute the $\langle p\,p\rangle$ correlator as

$$\begin{aligned}\langle{:}p(t)\,p(0){:}\rangle = -m^2\,\partial_t^2\langle{:}x(t)\,x(0){:}\rangle &= \frac{\alpha'\,\beta^2\,m^2}{\pi^2\,\ell^2}\int_0^\infty d\omega\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}\,\frac{\omega\cos\omega t}{e^{\beta\omega}-1}\\ &= \frac{\alpha'\,\beta\,m^2}{\pi\,\ell^2}\int_{-\infty}^\infty \frac{d\omega}{2\pi}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}\,\frac{\beta|\omega|\,e^{-i\omega t}}{e^{\beta|\omega|}-1}.\end{aligned} \tag{3.28}$$

To obtain the power spectrum defined in (2.16) for the momentum p we Fourier transform in time t to obtain

$$I_p^{\mathrm{n}}(\omega) = \frac{\alpha'\beta m^2}{\pi\ell^2}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}\,\frac{\beta|\omega|}{e^{\beta|\omega|}-1}. \tag{3.29}$$

Here, the superscript "n" is for remembering that this power spectrum was computed using the normal ordered correlator $\langle{:}p\,p{:}\rangle$. Then we can exploit the relation (2.19) between the power spectrum for the auto-correlation function and the momentum spectrum and the previously derived expression for $\mu(\omega)$, (3.23), to obtain the power spectrum for the random force R, which is nothing but the random force correlator $\kappa^{\mathrm{n}}(\omega)$:

$$\kappa^{\mathrm{n}}(\omega) = I_R^{\mathrm{n}}(\omega) = \frac{I_p^{\mathrm{n}}(\omega)}{|\mu(\omega)|^2} = \frac{4\pi\ell^2}{\alpha'\beta^3}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}\,\frac{\beta|\omega|}{e^{\beta|\omega|}-1}. \tag{3.30}$$

[20] The plus sign is because t is a lower index. If we raise t, this will have a minus sign, indicating a flow of energy toward the direction of the horizon (smaller r).

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Next, let us compute the physical time scales $t_{\rm relax}$ and $t_{\rm coll}$. First, from (3.23), one can compute the relaxation time $t_{\rm relax}$ defined in (2.20) as:

$$t_{\rm relax} = \mu(\omega = 0) \sim \frac{\alpha' \beta^2 m}{\ell^2}. \tag{3.31}$$

To compute the collision duration time $t_{\rm coll}$, we first need the real time auto-correlation function for the random force $\langle R\, R\rangle$:

$$\kappa^{\rm n}(t) = \langle : R(t) R(0) : \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, I_R^{\rm n}(\omega)\, e^{-i\omega t}. \tag{3.32}$$

By using the explicit form (3.30), we obtain

$$\kappa^{\rm n}(t) = \frac{2\,\ell^2}{\alpha'\,\beta^4} \left[\rho_c^2\, h_1(t,\beta) - (\rho_c^2 - 1)\, h_2(t,\beta,\rho_c)\right], \tag{3.33}$$

where we defined the functions

$$h_1(t,\beta) \equiv \int_{-\infty}^{\infty} dx\, \frac{|x|\, e^{-itx/\beta}}{e^{|x|} - 1}, \qquad h_2(t,\beta,\rho_c) \equiv \int_{-\infty}^{\infty} dx\, \frac{|x|\, e^{-itx/\beta}}{(1 + (\frac{x}{2\pi\rho_c})^2)(e^{|x|} - 1)}. \tag{3.34}$$

For $\rho_c \gg 1$, h_1 and h_2 are almost equal; if $x \ll \rho_c$, we can approximate $1 + (\frac{x}{2\pi\rho_c})^2$ in the integrand of h_2 by 1 while, if $x \gtrsim \rho_c \gg 1$, the integrand is almost vanishing because of the Bose-Einstein like factor $1/(e^{|x|} - 1)$. Therefore, the RR correlator evaluates to

$$\kappa^{\rm n}(t) \approx \frac{2\ell^2}{\alpha'\beta^4} h_1(t,\beta) = \frac{2\ell^2}{\alpha'\beta^4}\left[\left(\frac{\beta}{t}\right)^2 - \frac{\pi^2}{\sinh^2(\pi t/\beta)}\right]. \tag{3.35}$$

This function has a support of width of order β around $t = 0$. Therefore, using (2.22) we obtain the collision duration time

$$t_{\rm coll} \sim \beta = \frac{1}{T}, \tag{3.36}$$

The T dependence is as it should be from dimensional analysis in a CFT at temperature T, but the fact that this is independent of the 't Hooft coupling λ is not trivial.

The ratio of the two time scales is given by

$$\frac{t_{\rm relax}}{t_{\rm coll}} \sim \frac{\alpha' m}{\ell^2 T} \sim \frac{m}{\sqrt{\lambda}\, T}, \tag{3.37}$$

where we related α'/ℓ^2 to the boundary 't Hooft coupling[21] by using the relation $\ell^4/\alpha'^2 \sim \lambda$ [20, 24]. In the weak or moderate coupling regime, $\lambda \lesssim 1$, we can make this ratio large by considering a Brownian particle with $m \gg T$ and obtain the standard Brownian motion as explained below (2.23); the Brownian particle becomes thermalized only after numerous collisions with fluid particles. In the strong coupling regime, $\lambda \gg 1$, however, this is not the case and, in order to have the standard picture, we have to consider a much heavier

[21]Strictly speaking, this is only the 't Hooft coupling λ in the standard AdS_5 case, but we will use the same terminology to denote ℓ^4/α'^2 for other values of d as well.

Brownian particle with mass $m \gg \sqrt{\lambda}T$, which is always possible. On the other hand, if $T \ll m \ll \sqrt{\lambda}T$, the situation is totally different. Although the effect of a collision with a single fluid particle (with energy $\sim T$) is small, because the Brownian particle interacts with many fluid particles at the same time, it can become thermalized in a time much shorter than the time it takes for a single process of collision. To make this claim more quantitative, we estimate in appendix D the average time $t_{\rm mfp}$ between collisions. The contribution that a single collision makes to the random force $R(t)$ has width $t_{\rm coll}$. $R(t)$ consists of many such contributions, with the typical distance in time between two collisions being $t_{\rm mfp}$. Determining $t_{\rm mfp}$ is not entirely straightforward, as it requires us to analyze the four-point correlation function of the random force, and we only find a non-trivial answer once we take the fourth order correction to the Nambu-Goto action into account. As a result, $t_{\rm mfp}$ is suppressed by a factor of $1/\sqrt{\lambda}$ compared to $t_{\rm coll}$, the final result being

$$t_{\rm mfp} \sim \frac{1}{\sqrt{\lambda}T}. \tag{3.38}$$

At weak coupling $\lambda \ll 1$, we have $t_{\rm mfp} \gg t_{\rm coll}$ and the standard kinetic theory picture, where the Brownian particle is occasionally hit by a fluid particle, is valid. On the other hand, at strong coupling $\lambda \gg 1$, we have $t_{\rm mfp} \ll t_{\rm coll}$, namely, many collisions occur within the time scale for a single collision process to take place.[22] This supports the picture above that the Brownian particle interacts with many fluid particles at the same time.

4 Fluctuation-dissipation theorem

Thus far we have seen how the string probe in the bulk geometry holographically captures the Brownian motion of an external test particle introduced in the boundary CFT plasma. As we have seen explicitly, one can derive the Langevin equation for the string endpoint by tracing back the information about the part of the string that is touching the black hole and hence gets thermally excited due to the outgoing Hawking quanta. One of the hallmarks of non-equilibrium statistical mechanics is the fluctuation-dissipation theorem [32, 40] which relates the observables in the system perturbed infinitesimally away from equilibrium to equilibrium quantities. We now turn to show that not only are the results we derived in section 3 consistent with the fluctuation-dissipation theorem, but that we can in fact obtain this result directly from the gravity side.

4.1 Linear response theory

We begin our discussion of the fluctuation-dissipation theorem with a lightning review of linear response theory [32, 40].

Consider a system whose unperturbed Hamiltonian is given by H. Assume that, in the infinite past $t = -\infty$, the system was in an equilibrium state with the density matrix

$$\rho_e = \frac{e^{-\beta H}}{\operatorname{tr} e^{-\beta H}}. \tag{4.1}$$

[22]Perhaps the term "mean free path time" is not an appropriate one in this regime where a second collision takes place before the first one ends, and thus the particle is never freely moving. However, there being no other choice, we will continue to use this term in the strongly coupled regime.

Now perturb the system by adding an external force $K(t)$ conjugate to a quantity A. The total Hamiltonian is

$$H_{\rm tot} = H + H_{\rm ext}(t) = H - A\,K(t). \tag{4.2}$$

Under this perturbation, the change in another quantity B is given, to the first order in the perturbation $H_{\rm ext}$, by the so-called Kubo formula:

$$\Delta B(t) = \int_{-\infty}^{t} dt'\,K(t')\,\phi_{BA}(t-t'), \qquad \phi_{BA}(t) \equiv -i\langle [A(0), B(t)]\rangle, \tag{4.3}$$

where we defined $\langle \mathcal{O}\rangle \equiv \mathrm{tr}(\rho_e \mathcal{O})$ and $\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt}$. The function $\phi_{BA}(t)$ is called the response function.

If we consider a periodic force with frequency ω,

$$K(t) = K_0\,e^{-i\omega t}, \tag{4.4}$$

then (4.3) gives the following change in B:

$$\Delta B(t) = \mu_{BA}(\omega)\,K_0\,e^{-i\omega t}, \tag{4.5}$$

where the admittance $\mu_{BA}(\omega)$ is given by

$$\mu_{BA}(\omega) = \int_0^\infty dt\,\phi_{BA}(t)\,e^{i\omega t} = \frac{1}{i}\int_0^\infty dt\,\langle [A(0), B(t)]\rangle e^{i\omega t} = \beta \int_0^\infty dt\,\langle \dot{A}(0); B(t)\rangle e^{i\omega t}, \tag{4.6}$$

and with the *canonical correlator* $\langle X; Y\rangle$ defined by

$$\langle X; Y\rangle = \frac{1}{\beta}\int_0^\beta d\lambda\,\langle e^{\lambda H} X e^{-\lambda H} Y\rangle = \frac{1}{\beta}\int_0^\beta d\lambda\,\langle X(-i\lambda)\,Y\rangle\,, \tag{4.7}$$

which satisfies the following properties

$$\langle X(0); Y(t)\rangle = \langle Y(t); X(0)\rangle = \langle Y(0); X(-t)\rangle. \tag{4.8}$$

The relation (4.6) is called the fluctuation-dissipation theorem, because the right hand side is the fluctuation (correlator) in the equilibrium state ρ_e, while the left hand side yields the admittance which is related to the dissipation (friction).

In the case of Brownian motion, we can take $A = x$ and $H_{\rm ext} = -xK(t)$, where $K(t)$ is identified with the external force appearing in the Langevin equation (2.8). Then, for $B = p$, we obtain the admittance

$$\mu(\omega) = \frac{\beta}{m}\int_0^\infty dt\,\langle p(0); p(t)\rangle\,e^{i\omega t}. \tag{4.9}$$

Due to the relations (4.8), this implies

$$2\,\mathrm{Re}\,\mu(\omega) = \frac{\beta}{m}\int_{-\infty}^\infty dt\,\langle p(0); p(t)\rangle\,e^{i\omega t} = \frac{\beta}{m}\,I_p^c(\omega), \tag{4.10}$$

where $I_p^{\rm c}(\omega)$ is the power spectrum for p defined using the canonical correlator. From this, using the relation (2.19), one can derive a more direct relation between the friction and random force as

$$2\,{\rm Re}\,\gamma(\omega) = \frac{\beta}{m}\, I_R^{\rm c}(\omega) = \frac{\beta}{m}\,\kappa^{\rm c}(\omega), \tag{4.11}$$

which is sometimes called the second fluctuation-dissipation theorem, in contrast with (4.9) or (4.10) which is sometimes called the first fluctuation-dissipation theorem [32, 40].

4.2 Explicit check of fluctuation-dissipation theorem

The fluctuation-dissipation relations (4.9), (4.10), and (4.11) for Brownian motion were derived from the field theory viewpoint and are not immediately obvious from the bulk viewpoint. Here, let us explicitly check that they indeed hold using the explicit results obtained from the bulk in section 3.

Similarly to (3.2) or (3.4), we can compute the canonical correlator for x as

$$\langle x(0); x(t)\rangle = \frac{\alpha'\beta}{\pi\ell^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{1}{\omega^2}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}e^{-i\omega t}. \tag{4.12}$$

Because $p = m\dot{x}$, this implies the following canonical correlator for p:

$$\langle p(0); p(t)\rangle = \frac{\alpha'\beta m^2}{\pi\ell^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}e^{-i\omega t}. \tag{4.13}$$

This means that the power spectrum for p is

$$I_p^{\rm c}(\omega) = \frac{\alpha'\beta m^2}{\pi\ell^2}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}. \tag{4.14}$$

On the other hand, from (3.23), we immediately obtain

$$2\,{\rm Re}\,\mu(\omega) = \frac{\alpha'\beta^2 m}{\pi\ell^2}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}. \tag{4.15}$$

By comparing (4.14) and (4.15), we see that the fluctuation-dissipation theorem of the form (4.10) indeed holds. This also implies the second fluctuation-dissipation theorem (4.11).

These relations can be regarded as providing evidence that the motion of the string endpoint in the bulk can be described by a generalized Langevin equation. Note, in particular, that the way we derived the fluctuation (correlator) and the way we derived dissipation (admittance) were very different; for the former, we assumed thermal Hawking radiation near the horizon and measured the position of the string endpoint at the UV cut-off while, for the latter, we considered forced motion imposing a boundary condition at the horizon which was essentially the purely ingoing boundary condition. In the next subsection we describe how these two quantities are related directly from the bulk point of view. However, it would be desirable to have a more intuitive physical understanding of why this should be the case.

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4.3 Bulk proof of fluctuation-dissipation theorem

In subsection 4.2, we demonstrated that the fluctuation-dissipation relations holds for the special case of string probes in the BTZ spacetime by an explicit bulk computation. We now prove that the fluctuation-dissipation relations hold more generally, again from the bulk viewpoint.

Consider a string probe in the d-dimensional metric (2.26). We would like to turn on an electric field $F_{It} = E_I(t)$ on the flavor D-brane at $r = r_c$ and consider the resulting position $x^I(t) = X^I(t, r_c)$ of the string endpoint in response to it. If we take $A_t = E_I(t)\, X^I$, $A_I = 0$, then the boundary action, (3.15), can be written as

$$S_{\text{bdy}} = \int dt\, E_I(t) X^I = \int dt\, dr\, \delta(r - r_c) E_I(t) X^I. \tag{4.16}$$

This can be regarded a source term for the field X^I; upon inclusion of this term, the equation of motion (2.28) is changed to

$$\nabla^\mu \left[G_{IJ}(x)\, \partial_\mu X^I(x)\right] = -\frac{2\pi\alpha'}{\sqrt{-g}}\, \delta(r - r_c)\, E_I(t). \tag{4.17}$$

As is standard, we can solve this by using the retarded propagator

$$D^{IJ}_{\text{ret}}(t, r|t', r') = \theta(t - t')\langle [X^I(t, r), X^J(t', r')]\rangle, \tag{4.18}$$

where $X^I(t, r)$ satisfies the equation of motion (2.28) (or equivalently (4.17) with the right hand set to zero) and can be expanded in modes as in (2.47). Namely,

$$\begin{aligned} X^I(t, r) &= \sum_{\omega>0} [u^I_\omega(t, r) a_\omega + u^I_\omega(t, r)^* a^\dagger_\omega], \\ [a_\omega, a_{\omega'}] &= [a^\dagger_\omega, a^\dagger_{\omega'}] = 0, \qquad [a_\omega, a^\dagger_{\omega'}] = \delta_{\omega\omega'}, \end{aligned} \tag{4.19}$$

where $\{u^I_\omega(t, r)\}$ is a normalized basis of solutions to (2.28). Note that, with (4.19), the commutator appearing in (4.18) is actually a c-number and D^{IJ}_{ret} is independent of the state with respect to which we take the expectation value. D^{IJ}_{ret} can be shown to satisfy

$$\nabla^\mu \left[G_{IJ}(x)\, \partial_\mu D^{JK}_{\text{ret}}(t, r|t', r')\right] = i\, \frac{2\pi\alpha'}{\sqrt{-g}}\, \delta^K_I\, \delta(t - t')\, \delta(r - r'). \tag{4.20}$$

Therefore, the solution to (4.17) can be written, using D^{IJ}_{ret}, as

$$\begin{aligned} X^I(t, r) &= i \int_{-\infty}^{\infty} dt'\, D^{IJ}_{\text{ret}}(t, r|t', r_c) E_J(t') \\ &= i \int_0^\infty dt''\, \langle [X^I(t, r), X^J(t - t'', r_c)]\rangle E_J(t - t''). \end{aligned} \tag{4.21}$$

By setting $r = r_c$, we obtain the position $x^I(t) = X^I(t, r_c)$ of the string endpoint in response to the external force $E_I(t)$ as:

$$x^I(t) = i \int_0^\infty dt''\, \langle [x^I(t), x^J(t - t'')]\rangle E_J(t - t''). \tag{4.22}$$

If the fluctuation-dissipation theorem is to hold, this must be equal to the $x^I(t)$ obtained by using the Kubo formula (4.3), identifying $x^I(t)$ with the position of the Brownian particle in the boundary. From the action (4.16), one reads off the external force appearing in (4.2) to be $AK = E_J x^J$. Then, by applying the Kubo formula (4.3) for $B = x^I$,

$$\begin{aligned} x^I(t) &= -i \int_{-\infty}^{t} dt'\, E_J(t') \langle [x^J(0), x^I(t-t')] \rangle \\ &= i \int_0^{\infty} dt''\, \langle [x^I(t''), x^J(0)] \rangle E_J(t-t''), \end{aligned} \tag{4.23}$$

where $t'' = t - t'$. Because the system is stationary, the expectation value is invariant under shift of time: $\langle [x^I(t''), x^J(0)] \rangle = \langle [x^I(t), x^J(t-t'')] \rangle$. Therefore, the bulk response (4.22) is the same as the expression (4.23) computed from the boundary Kubo formula. This implies that the fluctuation-dissipation relations (4.9), (4.10), (4.11) indeed hold.

5 General dimensions

Thus far we have considered the case of $d = 3$ dimensional asymptotically AdS spacetimes only, which had the advantage that the wave equation for the modes of the string was exactly solvable. For general d, this is no longer possible and we have to use approximate methods. In this section, we employ the low frequency approximation $\omega \ll T$ and briefly summarize how some of the results of the previous sections get modified in the case of asymptotically AdS$_d$ spacetimes with general d.

The starting point is the metric (2.24), with Hawking temperature given in (2.25). The tortoise coordinate r_* is defined via

$$dr_* = \frac{\ell^2}{r^2\, h(r)}\, dr. \tag{5.1}$$

If we define $\eta = \exp[2\pi i/(d-1)]$, an explicit expression for the tortoise coordinate is

$$r_* = \sum_{k=0}^{d-2} \frac{\ell^2}{(d-1)\, \eta^k\, r_H} \log\left(\frac{r}{r_H} - \eta^k\right). \tag{5.2}$$

The term with $k = 0$ shows that near the horizon, $r_* \sim \frac{\ell^2}{(d-1)\, r_H} \log\bigl(\frac{r}{r_H} - 1\bigr)$, and from (5.1) we also see that the behavior near infinity is

$$r_* \sim -\frac{\ell^2}{r}, \qquad r \to \infty. \tag{5.3}$$

The generalization of (2.32) to arbitrary d reads

$$-\,\partial_t^2 X + \frac{h(r)}{\ell^4}\, \partial_r [r^4\, h(r)\, \partial_r X] = 0. \tag{5.4}$$

As before, we will exploit the translational invariance along t to decompose modes in plane waves; for convenience consider solutions of the form

$$X(t, r) = e^{-i\omega t}\, r^{-1}\, \Phi_\omega(r)\,, \tag{5.5}$$

from which it follows that the functions $\Phi_\omega(r)$ satisfy the following equation

$$\left[\frac{\partial^2}{\partial r_*^2}+\omega^2-V(r)\right]\Phi_\omega(r)=0\,, \tag{5.6}$$

with

$$V(r)=\frac{1}{\ell^4}\,r^2\,h(r)\,\left[2\,h(r)+r\,h'(r)\right]. \tag{5.7}$$

The wave equation (5.6) can be thought of as a time-independent Schrödinger equation for a particle moving in potential $V(r)$.

As in section 3, we want to exploit the semiclassical physics of Hawking radiation to learn about the behaviour of the string endpoint on the boundary. Once again it is worth noting that the dynamics of the scalar field $X(t,r)$ is similar to a minimally coupled scalar field propagating in an asymptotically AdS_2 spacetime with an event horizon. We would like to compute the admittance for the Langevin equation in this case.

In order to redo the computation that led to (3.23) we need to find the solution of the wave equation (5.6) which is purely ingoing at the horizon $r=r_H$. Let us denote this particular solution of the wave equation by $X_\omega^-(r)$. It is not possible to obtain this for general frequencies ω and hence we employ a low frequency approximation $\omega\ll T$ and use the so-called matching technique. Here, we only write down the final result of the computations, relegating the details to appendix E. The solution that is purely ingoing at the horizon behaves near infinity as

$$X_\omega^-(\rho)=C^+\,X_C^+(\rho)+C^-\,X_C^-(\rho), \tag{5.8}$$

where

$$C^\pm=\frac{1}{2}\left(1\pm\frac{1}{\nu^2}+i\,b\,\nu\right),\qquad X_C^\pm(\rho)=\left(1\mp\frac{i\nu}{\rho}\right)e^{\pm i\,\nu/\rho}. \tag{5.9}$$

Here, b is a constant independent of ν, whose precise value is not relevant for our purpose. Also, as before, we defined dimensionless quantities

$$\rho\equiv\frac{r}{r_H},\qquad \nu\equiv\frac{\ell^2\,\omega}{r_H}. \tag{5.10}$$

In terms of these, the low frequency condition $\omega\ll T$ reads $\nu\ll 1$. Actually, this result (5.8) is valid only to leading order in the ν expansion.

By carefully redoing the calculation in subsection 3.2, one can show that there is the following relation between the ingoing mode and the admittance $\mu(\omega)$:

$$\mu(\omega)=\frac{1}{\gamma[\omega]-i\omega}=-\frac{i\,(d-1)^2\,\alpha'\,m\,\beta^2\,\nu}{8\pi\,\ell^2\,\rho_c^4}\,\frac{X_\omega^-(\rho_c)}{\partial_{\rho_c}X_\omega^-(\rho_c)}\,. \tag{5.11}$$

Using the explicit expression of X_ω^- (eqs. (5.9), (5.8)), the final result is

$$\mu(\omega)=\frac{i(d-1)^2\,\alpha'\,m\,\beta^2}{8\pi\,\ell^2\,\rho_c^2\,\nu}\,\frac{\left[(1+i\,b\,\nu)\,\nu\,\rho_c-i\right]\,\nu\,\cos\!\left(\frac{\nu}{\rho_c}\right)+i\,\left[\rho_c-i\,\nu^3\,(1+i\,b\,\nu)\right]\,\sin\!\left(\frac{\nu}{\rho_c}\right)}{(1+i\,b\,\nu)\,\nu^2\,\cos\!\left(\frac{\nu}{\rho_c}\right)+i\,\sin\!\left(\frac{\nu}{\rho_c}\right)}. \tag{5.12}$$

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As mentioned above, this result is valid only to leading order in the expansion in ν. For small ν, the right hand side of (5.12) behaves as

$$\mu(\omega) = \frac{(d-1)^2\, \alpha'\, \beta^2\, m}{8\pi\, \ell^2} + \mathcal{O}(\omega). \tag{5.13}$$

For $d=3$, this agrees with (3.23) for $\nu \to 0$. Furthermore, this agrees with the drag force result (3.11) for general d, because $\gamma_0 = \mu[0]^{-1}$.

Just as we did for the $d=3$ case in subsection 3.1, we could also compute $\kappa(\omega)$ for general d in the low frequency approximation. However, this is not necessary, because we can directly obtain $\kappa(\omega)$ from the fluctuation-dissipation theorem (4.11), whose validity we already demonstrated for all values of d in subsection 4.3.

6 Stretched horizon and Brownian motion

The main philosophy of the membrane paradigm [41] is that, as far as an observer staying outside a black hole horizon is concerned, physics can be effectively described by assuming that the objects outside the horizon are interacting with an imaginary membrane, which is endowed with physical properties, such as temperature and resistance, and is sitting just outside the mathematical horizon. In section 3, we assumed that the Brownian motion of the UV endpoint of a string was caused by the boundary condition we impose at the horizon — all ingoing modes are falling in without being reflected, while the outgoing modes are always thermally populated. A curious question then is whether this boundary condition can be reproduced, in the spirit of the membrane paradigm, by postulating some interaction of the string with a membrane at the stretched horizon just outside the actual horizon. The interaction necessarily assumes a stochastic character, so it is natural to expect it to be described by a sort of Langevin equation. For a schematic explanation, see figure 3. It must be noted that the physics of the stretched horizon has been discussed in the AdS/CFT context previously in [42–45] and more recently in [46] where there is a nice discussion regarding the dynamics of the stretched horizon and the universality of hydrodynamic coefficients. We will now turn to a derivation of the properties of the stretched horizon in section 6.1 and then proceed to ask whether we can learn anything about the microscopic structure of the stretched horizon in section 6.2.

6.1 Langevin equation on stretched horizon

Let us consider placing an imaginary "IR brane" near the horizon at $\rho_s = 1+2\,\epsilon$, $\epsilon \ll 1$ and assume that the string ends on it.[23] If we assume that a force is acting on the endpoint, the equation of motion for the endpoint is, just as in (3.17), given by

$$-\frac{2\, r_H^3\, \epsilon}{\pi\, \alpha'\, \ell^4}\, \partial_\rho X|_{\rho_s} = F_s^X, \tag{6.1}$$

[23] In reality a string that dips into the black hole will continue merrily past the horizon without any trouble; the quickest way to see this of course is to pass to coordinates that are regular on the horizon such as ingoing Eddington-Finkelstein or Kruskal coordinates. Here we are interested in mimicking the boundary conditions of the black hole and hence will postulate there to be an imaginary boundary in the IR at $\rho = \rho_s$.

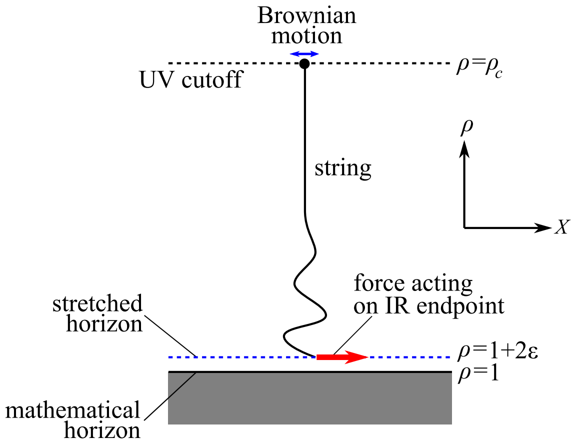

Figure 3. A membrane-paradigm like picture of the Brownian motion. There are friction and random force acting on the IR endpoint of the string on the stretched horizon, effectively giving the boundary condition.

where F_s^X is the force along the X direction measured with respect to the time t. Note that there is no term like $m\,\ddot{X}$ on the left hand side, because the endpoint has zero mass, having zero length. We assume that the force F_s^X, just as in the usual Langevin equation (2.8), has frictional and stochastic components:

$$F_s^X(t) = -\int_{-\infty}^{t} dt'\,\gamma_s(t-t')\,\partial_t X(t',\rho_s) + R_s(t), \tag{6.2}$$

$$\langle R_s(t)\rangle = 0, \qquad \langle R_s(t)R_s(t')\rangle = \kappa_s(t-t'), \tag{6.3}$$

where we allow the friction to depend on the past history through a memory kernel γ_s. We would like to choose γ_s and κ_s appropriately to reproduce the correct boundary condition described above.

Near the horizon the fluctuation of the string is given by (2.50)

$$X(t,\rho) = \sum_{\omega>0}\sqrt{\frac{\alpha'\,\beta}{2\,\ell^2\,\omega\,\log(1/\epsilon)}}\left[a_\omega^{(+)}\,e^{-i\omega(t-r_*)} + a_\omega^{(-)}\,e^{-i\omega(t+r_*)} + \text{h.c.}\right], \tag{6.4}$$

where ω is discretized with $\Delta\omega$ given in (2.45). $a_\omega^{(+)}$ and $a_\omega^{(-)}$ are annihilation operators for outgoing and ingoing modes, respectively. Depending on the boundary condition one imposes at the UV cut off, $a_\omega^{(+)}$ and $a_\omega^{(-)}$ get related to each other (for example, in the case of the Neumann boundary condition we imposed in subsection 2.3, they are related as $a_\omega^{(-)} = e^{i\,\theta_\omega}\,a_\omega^{(+)}$). However, because we are considering a Langevin equation which holds independent of such relations, we regard $a_\omega^{(+)}$ and $a_\omega^{(-)}$ as independent variables.

Plugging (6.4) in and going to the frequency space, we can write the equation of motion (6.1) as

$$-i\sqrt{\frac{\alpha'\,\beta\,\omega}{2\,\ell^2\,\log(1/\epsilon)}}\left[\left(\gamma_s[\omega]+\frac{r_H^2}{2\pi\,\alpha'\,\ell^2}\right)a_\omega^{(+)}\,e^{i\omega r_*}+\left(\gamma_s[\omega]-\frac{r_H^2}{2\pi\,\alpha'\,\ell^2}\right)a_\omega^{(-)}\,e^{-i\omega r_*}\right]=R_s(\omega) \tag{6.5}$$

for $\omega > 0$. Here, $\gamma_s[\omega]$ is the Fourier-Laplace transform of $\gamma_s(t)$ similar to (2.12) while $R_s(\omega)$ is the Fourier transform of $R_s(t)$ as in (2.11). In order to realize the boundary condition that all ingoing modes fall in without reflection, we should set the coefficient of $a_\omega^{(-)}$ to zero (since we want to be able to set the ingoing amplitude $a_\omega^{(-)}$ to any value). This gives

$$\gamma_s[\omega]=\frac{r_H^2}{2\pi\,\alpha'\,\ell^2}=\frac{2\pi\,\ell^2}{\alpha'\,\beta^2} \quad\Rightarrow\quad \gamma_s(t)=\frac{4\pi\,\ell^2}{\alpha'\,\beta^2}\,\delta(t). \tag{6.6}$$

Substituting this back into (6.5), we obtain the relation between the random force and the outgoing mode coefficients $a_\omega^{(+)}$ as

$$R_s(\omega)=-i\sqrt{\frac{8\pi^2\,\ell^2\,\omega}{\alpha'\,\beta^3\,\log(1/\epsilon)}}\,e^{i\omega r_*}\,a_\omega^{(+)}. \tag{6.7}$$

If this random force is to realize the thermal nature of the outgoing modes, $\langle a_\omega^{(+)\dagger} a_\omega^{(+)}\rangle = (e^{\beta\omega}-1)^{-1}$, then from (6.7) we obtain

$$\mathcal{D}(\omega)\,\langle R_s(\omega)^\dagger\,R_s(\omega)\rangle=\frac{2\,\ell^2\,\omega}{\alpha'\,\beta^2\,(e^{\beta\omega}-1)} \tag{6.8}$$

$$\approx\frac{2\,\ell^2}{\alpha'\,\beta^3}\,,\qquad \text{for } \beta\,\omega\ll 1\,. \tag{6.9}$$

Here $\mathcal{D}(\omega)$ is the density of states defined in (2.46). This means that the correlator for the random force is

$$\kappa_s(t-t')=\langle R(t)R(t')\rangle=\int_{-\infty}^{\infty}d\omega\,\mathcal{D}(\omega)\,\langle R_s(\omega)^\dagger\,R_s(\omega)\rangle\,e^{i\omega(t-t')}\approx\frac{4\pi\,\ell^2}{\alpha'\,\beta^3}\,\delta(t-t'). \tag{6.10}$$

The delta function behavior is due to the approximation we made in (6.9); the actual $\kappa_s(t-t')$ is nonvanishing for $|t-t'|\lesssim\beta$, as one can see if one uses the original exact expression (6.7), (6.8).

In summary, the boundary condition near the horizon can be effectively realized by a Langevin equation for the string endpoint $X(t,\rho_s)$ at the stretched horizon given by[24]

$$-\frac{2\,r_H^3\,\epsilon}{\pi\,\alpha'\,\ell^4}\,\partial_\rho X=-\frac{2\pi\,\ell^2}{\alpha'\,\beta^2}\,\partial_t X+R_s(t),\qquad \langle R_s(t)R_s(t')\rangle\approx\frac{4\pi\,\ell^2}{\alpha'\,\beta^3}\,\delta(t-t'). \tag{6.11}$$

The two terms on the right hand side of the first equation are, respectively, i) friction which precisely cancels the ingoing waves, and ii) random force which is responsible for the outgoing modes being thermally excited at the Hawking temperature.

[24]Note that these relations are operator relations whose full structure has been given in (6.7) and (6.8).

Given the auto-correlation function for the random force $R_s(t)$ acting on the string endpoint at the stretched horizon, we can exploit the Sutherland-Einstein relation (2.4) to compute the diffusion constant on the stretched horizon. We find

$$D^s_{\mathrm{AdS}_3} = \frac{2\,T^2}{\kappa_s(0)} = \frac{\alpha'}{2\,\pi\,\ell^2\,T} \tag{6.12}$$

which is the same as the diffusion constant for the string endpoint undergoing Brownian motion in the boundary (3.9). In deriving (6.12) we had to assume that the dynamics of the string endpoint on the stretched horizon obeys the Sutherland-Einstein relation derived for a point particle. In other words, we assumed that a point particle fixed on the stretched horizon will experience the same friction and random force as the ones appearing on the right hand side of (6.11), and thus will random walk with the diffusion constant (6.12). In fact, we will now argue that this is not quite unexpected from the viewpoint of the membrane paradigm.

In the context of the membrane paradigm, it is conventional to ascribe transport properties to the stretched horizon. In fact, it is well known that the shear viscosity of the black hole membrane saturates the famous bound derived in the boundary field theory, $\eta/s = 1/4\pi$, cf., [41, 47].[25] In the hydrodynamic regime of the AdS/CFT correspondence, ref. [46] argued that one can derive the universality of this ratio using the membrane paradigm, i.e., the physics of the stretched horizon similar to the discussion given above. We have here focussed on the stochastic Langevin process and derived the features of the membrane that reproduce the physics of strings impinging on the black hole. Again we see that the diffusion coefficient of heavy quarks in the boundary (3.9) agrees with that derived for the stretched horizon (6.12).

6.2 Granular structure on the stretched horizon

In [49, 50], Susskind and collaborators put forward a provocative conjecture that a black hole is made of a fundamental string covering the entire horizon. Although this picture must be somewhat modified [51] since we now know that branes are essential ingredients of string theory, it is still an attractive idea that, in the near horizon region where the local temperature becomes string scale, a stringy "soup" or "cloud" of strings and branes is floating around, covering the entire horizon.

If this picture is true, one naturally expects that it is this stringy cloud that is exerting frictional and stochastic forces on the IR endpoint of the fundamental string as described by (6.11); see figure 4. Can we learn anything about this stringy cloud? The stretched horizon is located a distance $\sim l_s = \sqrt{\alpha'}$ away from the mathematical horizon [50]. It is occupied by a string of length $L \sim S\, l_s$, with S the entropy of the black hole. If we associate one degree of freedom to each string segment of length l_s, the number of degrees of freedom equals the entropy, and one can try to think of these degrees of freedom in terms of free quasi-particles. The average separation between the quasi-particles is equal to

$$\Delta X \sim \frac{\ell}{r_H}\, l_{p(d)} \tag{6.13}$$

[25] See also [48] for another membrane paradigm inspired perspective on the ratio η/s.

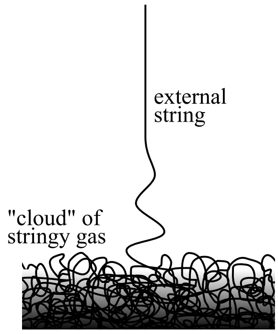

Figure 4. A possible microscopic picture of a black hole, where the horizon is covered by a stringy "cloud" made of strings and branes. An external fundamental string ending on a horizon is dissolved into the cloud and incessantly kicked around by the cloud.

with $l_{p(d)}$ the d-dimensional Planck length. If the quasi-particles move with the speed of light and σ represents the probability that quasi-particles will interact with the endpoint of the string, then we expect a mean free path time of the order of

$$t_{\rm mfp} \sim \frac{\Delta X}{\sigma}. \tag{6.14}$$

Supposing for a moment we assume that this is the same as the mean free path time on the boundary which, in appendix D, we argued to be given by $t_{\rm mfp} \sim 1/(T\sqrt{\lambda})$. Combined with (6.14) and (6.13) this leads to the interaction probability

$$\sigma \sim \frac{d-1}{4\pi}\,\frac{\ell}{l_s^2}\,l_{d(p)} \tag{6.15}$$

which for the usual AdS$_5$ case leads to a scaling with g_s and N as $\sigma \sim g_s^{1/2} N^{-1/6}$. This is a rather peculiar prediction for the interaction strength of the string endpoint with the quasi-particles. Since the quasi-particles are made out of strings (or branes) some g_s dependence is to be expected, and the interaction strength indeed vanishes as $g_s \to 0$.

In deriving (6.15) we have assumed that the mean free path time on the stretched horizon is identical to that on the boundary. This however, is unlikely to pertain as we explain now. In fact, we will argue that on general grounds we should expect that $t_{\rm mfp} \sim 1/T$ on any stretched horizon. The logic relies on using dimensional analysis coupled with thermal physics of black holes. Generically we expect,

$$t_{\rm mfp} = \frac{1}{T}\,\mathcal{G}\left(T l_s, \frac{l_{p(d)}}{l_s}, \frac{\ell}{l_s}\right) \tag{6.16}$$

where $\mathcal{G}$ is a function of the dimensionless ratios of the length scales available. We have fixed the overall normalization to be determined by the thermal scale on physical grounds. Furthermore, using the facts that: (i) the dynamics of string probes generically are unaware of the Planck scale (to determine which we could for instance use D-brane probes [52]) and, (ii) the geometry near a black hole horizon is the Rindler spacetime, which is insensitive to

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the cosmological constant, we can argue that $\mathcal{G} \sim 1$, i.e., it is independent of the hierarchy between the various length scales in the problem. More precisely, in the near horizon region, $\frac{r-r_H}{r_H} \ll 1$, the AdS$_d$ black hole metric (2.24) reduces to the Rindler metric:

$$ds_d^2 \approx -\tilde{r}^2 d\tilde{t}^2 + d\tilde{r}^2 + d\tilde{\vec{X}}_{d-2}^2, \tag{6.17}$$

where

$$\tilde{t} = 2\pi T t, \qquad \tilde{r} = \sqrt{\frac{r - r_H}{\pi T}}, \qquad \tilde{\vec{X}}_{d-2} = \frac{r_H}{\ell} \vec{X}_{d-2}. \tag{6.18}$$

Because the metric (6.17) does not contain any scale such as ℓ, the dynamics of a fundamental string in the near horizon region can only depend on l_s (dependence on $l_{p(d)}$ is excluded as in (i)). Therefore, the mean free path time $\tilde{t}_{\rm mfp}$ determined from the dynamics of the string can only depend on l_s. However, because $\tilde{t}_{\rm mfp}$ is dimensionless, it should be that $\tilde{t}_{\rm mfp} \sim 1$, which means $t_{\rm mfp} \sim 1/T$. One can give a more concrete argument by using the argument in appendix D applied to the near-horizon geometry (6.17).

Now using $t_{\rm mfp} \sim 1/T$ we can conclude that the interaction probability only depends on the ratio of the d-dimensional Planck scale and string scale:

$$\sigma \sim \frac{d-1}{4\pi} \frac{l_{p(d)}}{l_s}, \tag{6.19}$$

which suggests a universal dynamics of the stretched horizon independent of the asymptotics of the spacetime. Nevertheless (6.19) leads to an interaction probability which is a non-trivial function of g_s as $l_{p(d)}$ depends non-trivially on the details of the compactification.[26]

Clearly, it would be interesting to explore this line of thought further and, for example, also find an interpretation for the collision time. However, many of the assumptions we made are highly questionable. For example, we ignored backreaction, and only used the quadratic part of the Nambu-Goto action. The latter approximation certainly breaks down once we are a proper distance $\sim l_s$ away from the horizon. It is also unclear to what extent we can really think of the stretched horizon as a gas of almost free quasi-particles. We leave an exploration of these issues to future work.

7 Discussion

In this paper, we discussed Brownian motion in the holographic context, in order to shed light on near-equilibrium dynamics of strongly coupled thermal gauge theories. A useful probe exhibiting Brownian motion consists of a fundamental string stretching between the boundary and the horizon and being randomly excited by the black hole environment. We

[26] Actually, there is no consensus on where to place the stretched horizon. For example, refs. [53, 54] explained some thermodynamical properties of black holes by postulating the existence of quasi-particles living on a stretched horizon a distance $l_{p(d)}$ away from the horizon, instead of l_s. In this case with a stretched horizon located $l_{p(d)}$ away from the horizon, we obtain a simpler result $\sigma \sim 1$ instead of (6.19). This simple form of σ is appealing, but we do not know of a physical reason to choose one over the other.

established the relation between the observables associated with such Brownian particle in the boundary theory and those of the transverse mode excitations of the fundamental string. At the semiclassical level, the modes on the string are thermally excited due to Hawking radiation and, consequently, the motion of the boundary Brownian particle is described by a Langevin equation, which involves stochastic force and friction. In the bulk, the stochastic force corresponds to the random excitation of the string by the Hawking radiation, while the friction corresponds to the fact that the excitations on the string get dissipated into the horizon.

Although in this paper we focused on the relation at the semiclassical level between the boundary Brownian motion and the dynamics of the fundamental string in the bulk, the boundary-bulk dictionary we wrote down in subsection 2.4 in principle allows one to predict the precise correlations of the Hawking radiation quanta beyond the semiclassical approximation, in terms of the precise correlation functions for the boundary Brownian particle. Obtaining the latter of course requires one to compute correlation functions in strongly coupled plasmas, which is a difficult task. Nevertheless, such a dictionary is an important step toward understanding the microphysics underlying the fluid-gravity correspondence.

One of the particularly interesting results of the current paper is the estimate in subsection 3.3 for the time scales associated with the Brownian particle immersed in a CFT plasma:

$$t_{\rm relax} \sim \frac{m}{\sqrt{\lambda}\,T^2}, \qquad t_{\rm coll} \sim \frac{1}{T}, \qquad t_{\rm mfp} \sim \frac{1}{\sqrt{\lambda}\,T}. \tag{7.1}$$

Note that setting $m = T$ in $t_{\rm relax}$ gives $t_{\rm mfp}$, which is a consistency check because a fluid particle can be thought of as a Brownian particle with mass $\sim T$. The fact that $t_{\rm coll} \gg t_{\rm mfp}$ at strong coupling $\lambda \gg 1$ implies that a Brownian particle interacts with many plasma particles simultaneously. Because of this, a Brownian particle with mass $m \ll \sqrt{\lambda}\,T$ can thermalize in a time much shorter than $t_{\rm coll}$, the time elapsed in a single process of collision. This is reminiscent of the recent conjecture [55, 56] that black holes can scramble information very fast, whose dual picture is that a degree of freedom in the boundary theory interacts with a huge number of other degrees of freedom simultaneously. It would be interesting to study this possible connection further.

Historically, the main achievement of the theory of Brownian motion was the determination of the value of the Avogadro constant $\mathcal{N}_A = 6 \times 10^{23}\,\mathrm{mol}^{-1}$, which is huge but finite. If $\mathcal{N}_A$ were infinite, the diffusion constant would be zero and we would not be able to observe Brownian motion. The fact that we can observe it in nature gives evidence that $\mathcal{N}_A$ is finite and fluids are not continuous but made of molecules. Then, what is the analogue of the Avogadro constant in the Brownian motion in the AdS/CFT context we studied, and what is the bulk significance of it? In the case of $\mathrm{AdS}_5/\mathrm{SYM}_4$, the macroscopic energy density of the plasma scales as $E = \mathcal{O}(N^2)$, while the energy carried by a microscopic quantum is of the order of the temperature $T = \mathcal{O}(N^0)$. What corresponds to the Avogadro constant is the ratio of these, $N^2/N^0 = N^2$. The finiteness of $\mathcal{N}_A$ corresponds to the finiteness of N. In the bulk, on the other hand, what corresponds to E is the mass of the black hole, $M \sim R_s/G_N \sim \mathcal{O}(G_N^{-1})$ with R_s the Schwarzschild radius, while T is the Hawking temperature $T_H \sim \mathcal{O}(G_N^0)$. The ratio is $M/T_H = \mathcal{O}(G_N^{-1}) = \mathcal{O}(N^2)$. So, in the bulk, the finiteness of $\mathcal{N}_A$ corresponds to the finiteness of G_N, or to the fact that the energy carried by a Hawking radiation quantum is finite although it is much smaller than the mass of the black hole.

We considered non-relativistic Brownian motion in the current paper, which is the result of the quadratic approximation we made in (2.27) to the Nambu-Goto action. It would be interesting to generalize our treatment to the relativistic case, where Brownian motion and its Langevin dynamics are not very well understood; for a recent discussion, see e.g. [5]. Such a generalization can also be regarded as a generalization of the drag force computations of [20, 22], which are relativistic because the full Nambu-Goto action was taken into account, to non-stationary ($\omega \neq 0$) solutions. Also, with such a relativistic formalism, one can presumably give a more rigorous derivation of t_{mfp} than the one we did in appendix D.

As explained above, the stochastic force appearing in the Langevin equation is related to the friction term via the fluctuation-dissipation theorem. In the bulk the latter is mimicked by the dissipative nature of the event horizon, which is present for all black holes. On the other hand, the stochastic term arises due to the Hawking temperature of the black hole; yet only non-extremal black holes have finite Bekenstein-Hawking temperature. This leads to the naive puzzle that whereas the dissipation is always present, fluctuation is seemingly absent for extremal black holes since these have zero temperature. It would be interesting to see whether the quantum fluctuations which are present even at zero temperature suffice to account for the origin of the stochastic processes. Note that this is not a-priori unreasonable in the AdS/CFT context; although quantum processes are $1/N$ suppressed in the large N field theory, we had to account for the semi-classical Hawking radiation phenomena to see the origin of the random force in the Langevin equation. Furthermore, extremal black holes could also be subject to super-radiant type instabilities which can effectively mimic the physics of Hawking radiation. In fact, this feature has been exploited recently to show how the microstate 'geometries' can reproduce some features of the thermal Hawking spectrum [57].

The stochastic random force which drives the long time diffusive motion has a characteristic dependence on the temperature, which we derived assuming that the system was thermodynamically stable. As is well known, considering the global as opposed to the Poincaré patch of AdS provides two distinct black hole solutions at the same temperature — the small black hole which has negative specific heat and a large black hole which is in thermal equilibrium with the Hawking radiation. To be able to access both these solutions simultaneously one has to work in the global AdS geometry which has a compact spatial boundary. The physics of a probe string endpoint in the small black hole background should exhibit marked differences from the Brownian motion discussed above, despite the system experiencing the same temperature. In a finite volume system we naively expect the Brownian process to saturate after the time scale $t = \frac{\pi^3 \ell^4 T}{\alpha'}$, for in this time the particle has diffused throughout the system. In the bulk this presumably corresponds to the string diffusing out completely on the stretched horizon and becoming indistinguishable from the thermal atmosphere. This can in fact be used to probe the difference between the large and the small black hole. Imagine we normalize the physics of the string endpoint on the boundary to correspond to the Brownian motion undertaken in the large black hole. Using this as the UV boundary condition for the probe string in the small black hole background, we can examine the dynamics of the endpoint at the IR stretched horizon. A plausible conjecture for this dynamics is that the fluctuations of the string are macroscopically large on the stretched horizon, in fact will have a scale comparable to the black hole itself.

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A Normalized basis

In this appendix, we discuss the quantization of the action (2.27) obtained from the Nambu-Goto action and derive the normalized basis of solutions (2.48) to the equation of motion.

A.1 Canonical commutation relations and normalized basis

The canonical commutation relations for the theory (2.27):

$$S_{\rm NG}^{(2)} = -\frac{1}{4\pi\alpha'}\int d^2x\sqrt{-g(x)}\,g^{\mu\nu}(x)G_{IJ}(x)\frac{\partial X^I}{\partial x^\mu}\frac{\partial X^J}{\partial x^\nu}, \qquad x^\mu = t, r, \tag{A.1}$$

are given by

$$\begin{aligned} [X^I(x), X^J(x')]_\Sigma = 0, \qquad [X^I(x), n^\mu\partial_\mu X^J(x')]_\Sigma = i\frac{2\pi\alpha'}{\sqrt{h}}G^{IJ}\delta(r-r'), \\ [n^\mu\partial_\mu X^I(x), n^\nu\partial_\nu X^J(x')]_\Sigma = 0. \end{aligned} \tag{A.2}$$

Here, Σ is a Cauchy surface in the $x^\mu = t, r$ part of the spacetime (2.26), h_{ij} is the metric on Σ induced from $g_{\mu\nu}$, and n^μ is the future-pointing unit normal to Σ. For functions $f^I(x), g^I(x)$ satisfying the equation of motion (2.28), we can define the following inner product:

$$(f, g)_\Sigma = -\frac{i}{2\pi\alpha'}\int_\Sigma dx\sqrt{h}\,n^\mu G_{IJ}(f^I\partial_\mu g^{J*} - \partial_\mu f^I\, g^{J*}). \tag{A.3}$$

It can be shown that this inner product is independent of the choice of Σ, just as the standard Klein-Gordon inner product [38]. This inner product satisfies

$$(f,g)^* = -(f^*,g^*) = (g,f), \tag{A.4}$$
$$(af_1+bf_2,g)^* = a(f_1,g)+b(f_2,g), \qquad (f,ag_1+bg_2)^* = a^*(f,g_1)+b^*(f,g_2). \tag{A.5}$$

It is not difficult to show that the canonical commutation relations (A.2) are equivalent to

$$[(f,X)_\Sigma,(g,X)_\Sigma]_\Sigma = (f,g^*)_\Sigma \quad \forall f,g \text{ satisfying the equation of motion (2.28)}. \tag{A.6}$$

Let $\{u^I_\alpha(x)\}$ be a basis of normalized functions satisfying the equation of motion (2.28) such that

$$(u_\alpha,u_\beta) = -(u^*_\alpha,u^*_\beta) = \delta_{\alpha\beta}, \qquad (u_\alpha,u^*_\beta)=0, \tag{A.7}$$

and expand X^I as

$$X^I(x) = \sum_\alpha \left[a_\alpha u^I_\alpha(x) + a^\dagger_\alpha u^I_\alpha(x)^*\right]. \tag{A.8}$$

Then one can readily show that the condition (A.6) implies

$$[a_\alpha,a_\beta] = [a^\dagger_\alpha,a^\dagger_\beta]=0, \qquad [a_\alpha,a^\dagger_\beta]=\delta_{\alpha\beta}. \tag{A.9}$$

A.2 Normalized basis for AdS$_3$

As shown in the main text, in the AdS$_3$ case, the solution to the equation of motion can be written as (see eq. (2.40))

$$u_\omega(t,\rho) = A\left[f^{(+)}_\omega(\rho)+Bf^{(-)}_\omega(\rho)\right]e^{-i\omega t}, \tag{A.10}$$

where B satisfies boundary conditions (2.41) at $\rho=\rho_c$ and (2.44) at $\rho=1+2\epsilon$. The inner product (A.3) for this solution is

$$(u_\omega,u_\omega) = \frac{2\omega\ell^2|A|^2}{\alpha'\beta}\left[\frac{2\rho}{1+\rho^2\nu^2}+\log\left(\frac{\rho-1}{\rho+1}\right)\right]^{\rho=\rho_c}_{\rho=1+2\epsilon} \approx \frac{2\omega\ell^2|A|^2}{\alpha'\beta}\log\left(\frac{1}{\epsilon}\right). \tag{A.11}$$

From this, one obtains the normalized basis:

$$u_\omega(t,\rho) = \sqrt{\frac{\alpha'\beta}{2\ell^2\omega\log(1/\epsilon)}}\left[f^{(+)}_\omega(\rho)+Bf^{(-)}_\omega(\rho)\right]e^{-i\omega t}. \tag{A.12}$$

B Evaluation of displacement squared $s^2_{\mathrm{reg}}(t)$

In this appendix, we evaluate the displacement squared (3.5), which can be written as:

$$s^2_{\mathrm{reg}}(t) = \frac{4\alpha'\beta^2}{\pi^2\ell^2}\int_0^\infty \frac{d\nu}{\nu}\,\frac{1+\nu^2}{1+\rho_c^2\nu^2}\,\frac{\sin^2\frac{\pi t\nu}{\beta}}{e^{2\pi\nu}-1} = \frac{\alpha'\beta^2}{\pi^2\ell^2}\left(\frac{\rho_c^2-1}{\rho_c^2}I_1+\frac{1}{\rho_c^2}I_2\right), \tag{B.1}$$

where

$$I_1 = 4\int_0^\infty \frac{dx}{x(1+a^2x^2)}\frac{\sin^2\frac{kx}{2}}{e^x-1} = \int_{-\infty}^\infty \frac{dx}{|x|(1+a^2x^2)}\frac{1-e^{ikx}}{e^{|x|}-1},$$
$$I_2 = 4\int_0^\infty \frac{dx}{x}\frac{\sin^2\frac{kx}{2}}{e^x-1} = \int_{-\infty}^\infty \frac{dx}{|x|}\frac{1-e^{ikx}}{e^{|x|}-1}, \tag{B.2}$$

and we defined new variables by

$$x = 2\pi\nu, \qquad a = \frac{\rho_c}{2\pi}, \qquad k = \frac{t}{\beta}. \tag{B.3}$$

The integrals (B.2) can be evaluated using the standard method of deforming the contour on the complex x plane. For that, one first replaces $|x|$ with $\sqrt{x^2+\epsilon^2}$ with ϵ a small positive number. If $k>0$, one can then deform the contour to run vertically around the branch cut between $i\epsilon$ and ∞. The resulting integral is simpler than (B.2) and, after taking $\epsilon\to 0$, can be analytically evaluated. One should also take into account the contribution from the poles of the integrand on the imaginary axis. The final result is

$$\begin{aligned}I_1 &= \frac{1}{2}\left[e^{k/a}\mathrm{Ei}\left(-\frac{k}{a}\right)+e^{-k/a}\mathrm{Ei}\left(\frac{k}{a}\right)\right]+\frac{1}{2}\left[\psi\left(1+\frac{1}{2\pi a}\right)+\psi\left(1-\frac{1}{2\pi a}\right)\right]\\ &\quad+\frac{e^{-2\pi|k|}}{2}\left[\frac{{}_2F_1(1,1+\frac{1}{2\pi a};2+\frac{1}{2\pi a};e^{-2\pi|k|})}{1+\frac{1}{2\pi a}}+\frac{{}_2F_1(1,1-\frac{1}{2\pi a};2-\frac{1}{2\pi a};e^{-2\pi|k|})}{1-\frac{1}{2\pi a}}\right]\\ &\quad-\frac{\pi}{2}(1-e^{-|k|/a})\cot\frac{1}{2a}+\log\left(\frac{2a\sinh\pi k}{k}\right),\\ I_2 &= \log\left(\frac{\sinh\pi k}{\pi k}\right).\end{aligned} \tag{B.4}$$

where $\mathrm{Ei}(z)$ is the exponential integral, ${}_2F_1(\alpha,\beta;\gamma;z)$ is the hypergeometric function, and $\psi(z)=(d/dz)\log\Gamma(z)$ is the digamma function. For $\mathrm{Ei}(z)$, we take a branch where both $\mathrm{Ei}(x>0)$ and $\mathrm{Ei}(x<0)$ are real.

If $\rho_c \gg 1$ and thus $a\gg 1$, one can use the expressions (B.4) to derive the following behavior:

$$I_1 = \begin{cases}\frac{\pi k^2}{2a}+\mathcal{O}(a^{-2})\\ \pi k+\mathcal{O}(\log k)\end{cases} \qquad I_2 = \begin{cases}\mathcal{O}(a^0) & (k\ll a)\\ \pi k+\mathcal{O}(\log k) & (k\gg a)\end{cases} \tag{B.5}$$

Therefore, if $\rho_c\gg 1$, $s^2_{\mathrm{reg}}(t)$ has the following behavior:

$$s^2_{\mathrm{reg}}(t) = \begin{cases}\frac{\alpha'}{\ell^2\rho_c}t^2+\mathcal{O}(\frac{1}{\rho_c^2}) & (t\ll\beta)\\ \frac{\alpha'\beta}{\pi\ell^2}t+\mathcal{O}(\log\frac{t}{\beta}) & (t\gg\beta)\end{cases} \tag{B.6}$$

C Distribution of momentum p

In this appendix, we compute the probability distribution of the momentum $p=m\dot{x}$, where x is the position of the string endpoint at the UV cut-off $\rho=\rho_c$, and show that it is exactly equal to the Maxwell-Boltzmann distribution.[27]

[27]Here, we will ignore the fact that the mass of the quark gets corrected in thermal medium [20], and make a crude estimate by using the bare mass (2.39).

From (2.51), the momentum of the particle is

$$p = m\dot{x}(t) = -\frac{im}{\ell}\sum_{\omega>0}\sqrt{\frac{2\alpha'\beta\omega}{\log(1/\epsilon)}}\left[\frac{1-i\nu}{1-i\rho_c\nu}\left(\frac{\rho_c-1}{\rho_c+1}\right)^{i\nu/2}e^{-i\omega t}a_\omega - \text{h.c.}\right] \tag{C.1}$$

We would like to know the probability distribution $f(p)$ of p. By definition,

$$\langle e^{ip\xi}\rangle = \int_{-\infty}^{\infty} dp\, e^{ip\xi} f(p). \tag{C.2}$$

Namely, $f(p)$ is the Fourier transform of $\langle e^{ip\xi}\rangle$. So, what we want to compute is

$$\langle :e^{ip\xi}:\rangle = \left\langle :\exp\left\{\frac{\xi m}{\ell}\sum_{\omega>0}\sqrt{\frac{2\alpha'\beta\omega}{\log(1/\epsilon)}}\left[\frac{1-i\nu}{1-i\rho_c\nu}\left(\frac{\rho_c-1}{\rho_c+1}\right)^{i\nu/2}e^{-i\omega t}a_\omega - \text{h.c.}\right]\right\}:\right\rangle, \tag{C.3}$$

where we regularized the operator by normal ordering. The expectation value is with respect to the density matrix (2.52). Using the identity

$$\text{tr}\left[e^{-\beta\omega a^\dagger a} :e^{\alpha a - \alpha^* a^\dagger}:\right] = \frac{1}{1-e^{-\beta\omega}}\exp\Big(-\frac{|\alpha|^2}{e^{\beta\omega}-1}\Big), \tag{C.4}$$

we can compute

$$\langle :e^{ip\xi}:\rangle = C\exp\left[-\frac{2\xi^2\alpha' m^2}{\ell^2}\int_0^\infty \frac{d\nu\,\nu(1+\nu^2)}{(1+\rho_c^2\nu^2)(e^{2\pi\nu}-1)}\right], \tag{C.5}$$

where C is a constant independent of ξ and we rewrote the sum over ω by an integral using (2.45). This integral can be evaluated by deforming the contour in the complex plane, just as we did for (B.2), the result being

$$\int_0^\infty \frac{d\nu\,\nu(1+\nu^2)}{(1+\rho_c^2\nu^2)(e^{2\pi\nu}-1)} = \frac{\rho_c^2-1}{4\rho_c^4}\left[\pi\cot\frac{\pi}{\rho_c} - \psi\Big(1+\frac{1}{\rho_c}\Big) - \psi\Big(1-\frac{1}{\rho_c}\Big) - 2\log\rho_c\right] + \frac{1}{24\rho_c^2}, \tag{C.6}$$

where $\psi(z) = (d/dz)\log\Gamma(z)$ is the digamma function. Using this expression, it is easy to show that, for large ρ_c,

$$\int_0^\infty \frac{d\nu\,\nu(1+\nu^2)}{(1+\rho_c^2\nu^2)(e^{2\pi\nu}-1)} = \frac{1}{4\rho_c} + \cdots. \tag{C.7}$$

Therefore, from (C.5), we obtain

$$\langle :e^{ip\xi}:\rangle = Ce^{-m\xi^2/2\beta} \tag{C.8}$$

for large ρ_c, where we used (2.39). By Fourier transforming,

$$f(p) = \int_{-\infty}^{\infty}\frac{d\xi}{2\pi}e^{-ip\xi}\langle :e^{ip\xi}:\rangle \propto e^{-\beta E_p}, \qquad E_p \equiv \frac{p^2}{2m}. \tag{C.9}$$

This is exactly the Maxwell-Boltzmann distribution of particles with energy E_p. Therefore, for large ρ_c, the endpoint of the string behaves like a non-relativistic particle with mass m immersed in a thermal bath of temperature T.

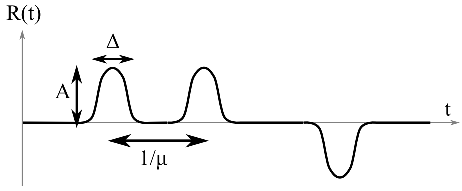

Figure 5. A sample of the stochastic variable $R(t)$, which consists of many pulses randomly distributed.

D Mean free path time t_{mfp}

In subsection 3.3, we discussed the time scales associated with Brownian motion: the relaxation time t_{relax} and the collision duration time t_{coll}. In this appendix, we evaluate t_{mfp}, the mean free path time, or the typical time between two collisions, using the correlators of the random force $R(t)$ in the case of AdS$_3$. We argue that the mean free path time is given by

$$t_{\text{mfp}} \sim \frac{1}{\sqrt{\lambda}T}, \tag{D.1}$$

where $\lambda \sim \ell^4/\alpha'^2$ is the 't Hooft coupling, although we are unable to give a rigorous derivation. We expect that this holds in more general cases, including AdS$_5$.

In subsection D.1, we discuss how to determine characteristic time scales from correlators in general. In subsection D.2, we compute t_{mfp} for the Brownian motion in the case of AdS$_3$.

D.1 Correlators and time scales

Consider a stochastic quantity $R(t)$ whose functional form consists of many pulses randomly distributed. Let the form of a single pulse be $f(t)$, with width Δ and amplitude A. Furthermore, assume that the pulses come with random signs. If we have k pulses at $t = t_i$ ($i = 1, 2, \ldots, k$), then $R(t)$ is given by

$$R(t) = \sum_{i=1}^{k} \epsilon_i f(t - t_i), \tag{D.2}$$

where $\epsilon_i = \pm 1$ are random signs. For a schematic picture, see figure 5. Let us assume that the distribution of pulses obeys the Poisson distribution. Namely, the probability that there are k pulses in an interval of length τ, say $[0, \tau]$, is given by

$$P_k(\tau) = e^{-\mu\tau} \frac{(\mu\tau)^k}{k!}. \tag{D.3}$$

Here, μ is the number of pulses per unit time. In other words, $1/\mu$ is the average distance between two pulses. We do not assume that the pulses are well separated; namely, we do *not* assume $\Delta \ll 1/\mu$. Later, we will identify $R(t)$ with the random force in the Langevin equation; the pulses are contributions from a collision with a fluid particle, and therefore $t_{\rm mfp} = 1/\mu$.

The 2-point function for R can be written as

$$\langle R(t)R(t')\rangle = \sum_{k=1}^{\infty} e^{-\mu\tau}\frac{(\mu\tau)^k}{k!}\sum_{i,j=1}^{k}\langle \epsilon_i\epsilon_j f(t-t_i)f(t'-t_j)\rangle_k, \tag{D.4}$$

where we assumed $t, t' \in [0,\tau]$ and $\langle\ \rangle_k$ is the statistical average when there are k pulses during $[0,\tau]$. Because k pulses are randomly and independently distributed in the interval $[0,\tau]$, this expectation value is computed as

$$\begin{aligned}\sum_{i,j=1}^{k}&\langle \epsilon_i\epsilon_j f(t-t_i)f(t'-t_j)\rangle_k \\ &= \frac{1}{\tau^k}\int_0^\tau dt_1\cdots dt_k\left[\sum_{i=1}^{k} f(t-t_i)f(t'-t_i) + \sum_{i\neq j}^{k}\langle\epsilon_i\epsilon_j\rangle_k f(t-t_i)f(t'-t_j)\right].\end{aligned} \tag{D.5}$$

Here, the second term vanishes because $\langle\epsilon_i\epsilon_j\rangle_k = 0$ for $i\neq j$. Therefore, one readily computes

$$\begin{aligned}\sum_{i,j=1}\langle \epsilon_i\epsilon_j f(t-t_i)f(t'-t_j)\rangle_k &= \frac{k}{\tau}\int_0^\tau dt_1 f(t-t_1)f(t'-t_1) \\ &\approx \frac{k}{\tau}\int_{-\infty}^{\infty} dt_1\, f(t-t'-t_1)f(-t_1) \equiv \frac{k}{\tau}F(t-t').\end{aligned} \tag{D.6}$$

Here, in going to the second line, we took τ to be much larger than the width Δ of $f(t)$, which is always possible because τ is arbitrary. Substituting this back into (D.4), we find

$$\langle R(t)R(t')\rangle = \mu F(t-t'). \tag{D.7}$$

In a similar way, one can compute the following 4-point function:

$$\langle R^2(t)R^2(t')\rangle = \sum_{k=1}^{\infty} e^{-\mu\tau}\frac{(\mu\tau)^k}{k!}\sum_{i,j,m,n=1}^{k}\langle \epsilon_i\epsilon_j\epsilon_m\epsilon_n f(t-t_i)f(t-t_j)f(t'-t_m)f(t'-t_n)\rangle_k. \tag{D.8}$$

Again, the expectation value $\langle\epsilon_i\epsilon_j\epsilon_m\epsilon_n\rangle_k$ vanishes unless some of i,j,m,n are equal. The possibilities are $i=j\neq m=n$, $i=m\neq j=n$, $i=n\neq j=m$, and $i=j=m=n$. Therefore,

$$\begin{aligned}\sum_{i,j,m,n=1}^{k}&\langle \epsilon_i\epsilon_j\epsilon_m\epsilon_n f(t-t_i)f(t-t_j)f(t'-t_m)f(t'-t_n)\rangle_k \\ &= \Big\langle \sum_{i\neq j}^{k}\left[f(t-t_i)^2 f(t'-t_j)^2 + 2f(t-t_i)f(t'-t_i)f(t-t_j)f(t'-t_j)\right] \\ &\qquad + \sum_{i=1}^{k} f(t-t_i)^2 f(t'-t_i)^2\Big\rangle_k\end{aligned}$$

$$= \frac{k(k-1)}{\tau^2} \int_{-\infty}^{\infty} dt_1 dt_2 \left[f(t-t_1)^2 f(t'-t_2)^2 + f(t-t_1) f(t'-t_1) f(t-t_2) f(t'-t_2) \right] + \frac{k}{\tau} \int_{-\infty}^{\infty} dt_1 f(t-t_1)^2 f(t'-t_1)^2. \qquad \text{(D.9)}$$

Substituting this back into (D.8), we obtain

$$\begin{aligned} \langle R^2(t) R^2(t') \rangle &= \mu^2 [F(0)^2 + F(t-t')^2] + \mu \int_{-\infty}^{\infty} du\, f(t-t'-u)^2 f(-u)^2 \\ &= \langle R^2(t) \rangle \langle R^2(t') \rangle + \langle R(t) R(t') \rangle^2 + \mu \int_{-\infty}^{\infty} du\, f(t-t'-u)^2 f(-u)^2. \end{aligned} \qquad \text{(D.10)}$$

For example, consider the following shape function for the pulse

$$f(t) = A e^{-t^2/2\Delta^2}. \qquad \text{(D.11)}$$

Then, one computes

$$\langle R(t) R(0) \rangle = \sqrt{\pi}\, \Delta\, \mu\, A^2 e^{-t^2/4\Delta^2}, \qquad \text{(D.12)}$$

$$\langle R^2(t) R^2(0) \rangle = \langle R^2(t) \rangle \langle R^2(0) \rangle + \langle R(t) R(0) \rangle^2 + \frac{1}{\sqrt{2\pi}\, \Delta\, \mu} \langle R(t) R(0) \rangle^2. \qquad \text{(D.13)}$$

Therefore, if we know the behavior of $\langle R(t)R(0)\rangle$, we can read off Δ and μA from (D.12). If we further know $\langle R^2(t)R^2(0)\rangle$ then, from (D.13), we can read off μ. In particular, if we denote the last term in (D.13) by $\langle R^4\rangle'$, then

$$\mu^{-1} \sim \frac{\langle R^4 \rangle'}{\langle R^2 \rangle^2} \Delta. \qquad \text{(D.14)}$$

This result (D.14) is expected to be true for other forms of $f(t)$, not just for the Gaussian case (D.11).

Note that the treatment above is classical. If $R(t)$ is a quantum operator, we should consider the classical part of the correlators by appropriately subtracting quantum divergences.

D.2 Evaluation of t_{mfp} for Brownian motion

Let us evaluate t_{mfp} for the boundary Brownian motion, by identifying the stochastic function $R(t)$ in the previous subsection with the random force appearing in the Langevin equation. A pulse $f(t)$ corresponds to the contribution from a collision with a single plasma particle. Δ is the time elapsed in a single collision, namely $\Delta = t_{\text{coll}}$, while $1/\mu$ is the time between two collisions, namely $1/\mu = t_{\text{mfp}}$.

Using eq. (2.51) as well as the relations $p = m\dot{x}$ and $R(\omega) = p(\omega)/\mu(\omega)$, where $\mu(\omega)$ is given by (3.23), we can write the random force $R(t)$ as

$$R(t) = \sum_{\omega>0} (r_\omega\, e^{i\omega t}\, a_\omega + \text{h.c.}), \qquad r_\omega = -i \sqrt{\frac{8\pi^2 \ell^2 \omega}{\alpha' \beta^3 \log(1/\epsilon)}} \frac{1-i\nu}{1-i\nu/\rho_c} \left(\frac{\rho_c - 1}{\rho_c + 1} \right)^{\frac{i\nu}{2}}. \qquad \text{(D.15)}$$

Because a_ω are free harmonic oscillators, it is easy to show that

$$\langle :R^2(t)R^2(0):\rangle = \langle :R^2(t):\rangle\langle :R^2(0):\rangle + \langle :R(t)R(0):\rangle^2. \tag{D.16}$$

Here, we are considering the normal-ordered correlators because the result of the previous subsection applies to the classical piece of correlators; henceforth, normal ordering of operators will be understood. By comparing (D.16) with (D.13), we appear to have $t_{\rm mfp} = 0$. However, this is due to the non-relativistic approximation we made in (2.27) when we expanded the Nambu-Goto action up to quadratic order. If we keep the next order (quartic) terms, we obtain the following additional contribution to the Hamiltonian:

$$H^{(4)} = -\frac{1}{16\pi\alpha'}\int_{r_s}^{r_c} dr \left[\frac{(\partial_t X)^2}{h(r)} - \frac{r^4 h(r)}{\ell^4}(\partial_r X)^2\right]^2, \qquad h(r) = 1 - \left(\frac{r_H}{r}\right)^2, \tag{D.17}$$

where $r_s = (1{+}2\epsilon)\, r_H$, $\epsilon \ll 1$. This corresponds to the first relativistic correction to the non-relativistic action $S^{(2)}_{\rm NG}$. In the presence of this interaction, there is an extra contribution to the correlator $\langle R^2(t)R^2(0)\rangle$ coming from the contractions with the terms in $H^{(4)}$. If we consider the case with $t = 0$, we have

$$\langle R^2(0)R^2(0)\rangle \equiv \langle R^4\rangle = 2\langle R^2\rangle_0^2 - \beta\langle R^4 H^{(4)}\rangle_0, \tag{D.18}$$

where $\langle\ \rangle_0$ is the expectation value with respect to the quadratic action $S^{(2)}_{\rm NG}$, i.e., it is the expectation value with respect to the density matrix (2.52).

So, let us evaluate the last term in (D.18), which will be denoted by $\langle R^4\rangle'$. Using the expansions (2.47) and (D.15), the explicit expression for $\langle R^4\rangle'$ is

$$\begin{aligned}\langle R^4\rangle' = -\frac{\beta}{16\pi\alpha'}\Bigg\langle &\sum_{\omega_1,\dots,\omega_4>0} (r_{\omega_1}a_{\omega_1} + \text{h.c.})(r_{\omega_2}a_{\omega_2} + \text{h.c.})(r_{\omega_3}a_{\omega_3} + \text{h.c.})(r_{\omega_4}a_{\omega_4} + \text{h.c.})\\ &\times \int_{r_s}^{r_c} dr \left\{\frac{1}{h}\Big[\sum_{\omega>0}\omega(u_\omega a_\omega - u^*_\omega a^\dagger_\omega)\Big]^2 + \frac{r^4 h}{\ell^4}\Big[\sum_{\omega>0}\big((\partial_r u_\omega)a_\omega + (\partial_r u^*_\omega)a^\dagger_\omega\big)\Big]^2\right\}^2\Bigg\rangle_0.\end{aligned} \tag{D.19}$$

There are many terms coming from the expansion of this. Let us focus on the following term in particular:

$$-\frac{\beta}{16\pi\alpha'}\sum_{\omega_1,\dots,\omega_4}\sum_{\omega'_1,\dots,\omega'_4}\omega_1\,\omega_2\,\omega_3\,\omega_4\, r^*_{\omega_1}r^*_{\omega_2}r^*_{\omega_3}r_{\omega_4}\langle a^\dagger_{\omega_1}a^\dagger_{\omega_2}a^\dagger_{\omega_3}a_{\omega_4}a_{\omega'_1}a_{\omega'_2}a_{\omega'_3}a^\dagger_{\omega'_4}\rangle_0\int_{r_s}^{r_c}\frac{dr}{h^2}u_{\omega'_1}u_{\omega'_2}u_{\omega'_3}u^*_{\omega'_4}. \tag{D.20}$$

There are various ways to contract $a, a^\dagger$. Let us take the term obtained by contracting a_{ω_i} against $a^\dagger_{\omega'_i}$, or $a^\dagger_{\omega_i}$ against $a_{\omega'_i}$, where $i = 1,\dots,4$. Other contractions give similar contributions. This particular contraction gives the following:

$$\begin{aligned}&-\frac{\beta}{16\pi\alpha'}\sum_{\omega_1,\dots,\omega_4}\omega_1\,\omega_2\,\omega_3\,\omega_4\, r^*_{\omega_1}r^*_{\omega_2}r^*_{\omega_3}r_{\omega_4}\left[\prod_{i=1}^4\frac{1}{e^{\beta\omega_i}-1}\right]\int_{r_s}^{r_c}\frac{dr}{h^2}u_{\omega_1}u_{\omega_2}u_{\omega_3}u^*_{\omega_4}\\ &\qquad\sim \frac{1}{\alpha'\beta^3}\sum_{\omega_1,\dots,\omega_4\lesssim\beta^{-1}} r^*_{\omega_1}r^*_{\omega_2}r^*_{\omega_3}r_{\omega_4}\int_{r_s}^{r_c}\frac{dr}{h^2}u_{\omega_1}u_{\omega_2}u_{\omega_3}u^*_{\omega_4},\end{aligned} \tag{D.21}$$

where the Bose-Einstein factor $1/(e^{\beta\omega_i}-1)$ has effectively cut off the ω_i sum at β^{-1}. From now on, we do not keep track of numerical factors.

We would like to evaluate the r integral in (D.21). Because the integrand in (D.21) has a second order pole at $r = r_H$ due to h^{-2}, the dominant contribution comes from $r \sim r_s \approx r_H$. For a while, let us instead consider the case where there is *a first order pole at* $r = r_H$*, by replacing* h^{-2} *by* h^{-1}. From (2.50) and (2.31), near $r = r_H$,

$$u_\omega \approx \sqrt{\frac{\alpha'\beta}{2\ell^2\omega\log(1/\epsilon)}}(e^{i\omega r_*}+e^{i\theta_\omega}e^{-i\omega r_*})e^{-i\omega t}, \qquad \frac{dr}{h} \approx \frac{r_H^2}{\ell^2}dr_* = \frac{4\pi^2\ell^2}{\beta^2}dr_*. \tag{D.22}$$

Therefore, the integral in (D.21) (with h^{-2} replaced by h^{-1}) is

$$\begin{aligned} &\sim \frac{\alpha'^2}{\ell^2[\log(1/\epsilon)]^2\sqrt{\omega_1\omega_2\omega_3\omega_4}}\int_{-\frac{\beta}{4\pi}\log(\frac{1}{\epsilon})} dr_*\,[e^{i\omega_1 r_*}+e^{i\theta_{\omega_1}}e^{-i\omega_1 r_*}][e^{i\omega_2 r_*}+e^{i\theta_{\omega_2}}e^{-i\omega_2 r_*}] \\ &\qquad\times[e^{i\omega_3 r_*}+e^{i\theta_{\omega_3}}e^{-i\omega_3 r_*}][e^{-i\omega_4 r_*}+e^{-i\theta_{\omega_4}}e^{i\omega_4 r_*}]\,e^{-i(\omega_1+\omega_2+\omega_3-\omega_4)t} \end{aligned} \tag{D.23}$$

The dominant part in the $\epsilon \to 0$ limit can be easily evaluated by noting that

$$\int_{-\frac{\beta}{4\pi}\log(\frac{1}{\epsilon})} dr_*\,e^{i\omega r_*} = \delta_{\omega,0}\,\frac{\beta}{4\pi}\log\left(\frac{1}{\epsilon}\right) + (\text{finite as } \epsilon\to 0). \tag{D.24}$$

For example, by collecting the first terms in the four pairs of the brackets in (D.23), one finds

$$\sim \frac{\alpha'^2\beta}{\ell^2\log(1/\epsilon)\sqrt{\omega_1\omega_2\omega_3\omega_4}}\,\delta_{\omega_1+\omega_2+\omega_3,\,\omega_4}. \tag{D.25}$$

Note that the finite part in (D.24) does not survive in the $\epsilon \to 0$ limit. If we plug this result back into (D.21), using the explicit expression for r_ω in (D.15), we find

$$\sim \frac{\ell^2}{\alpha'\beta^8[\log(1/\epsilon)]^3}\sum_{\omega_1,\omega_2,\omega_3\lesssim\beta^{-1}}\left[\frac{1-i\nu_1}{1-i\nu_1/\rho_c}\left(\frac{\rho_c-1}{\rho_c+1}\right)^{i\nu_1/2}\right]\Big[\,2\,\Big]\Big[\,3\,\Big]\Big[\,1+2+3\,\Big]^*. \tag{D.26}$$

Here, "[2]" denotes the previous factor with ν_1 replaced by ν_2. "[3]" and "$[\,1+2+3\,]$" are similar. By rewriting the sum by integral using (2.45) and using the fact that $\rho_c \gg 1$, this is estimated as

$$\sim \frac{\ell^2}{\alpha'\beta^5}\int_{\lesssim\beta^{-1}} d\omega_1\,d\omega_2\,d\omega_3 \sim \frac{\ell^2}{\alpha'\beta^8} \sim \frac{1}{\beta^8\sqrt{\lambda}}, \tag{D.27}$$

where we used the relation $\lambda \sim \ell^4/\alpha'^2$. There are many other terms we did not discuss, such as other contractions of (D.20), but these will not affect this estimate.

However, of course, this is not precisely what we wanted to evaluate; we have replaced h^{-2} in (D.21) by h^{-1}. However, using h^{-2} instead will change the above discussion, because the r integral around $r = r_H$ will now give a power ($\sim 1/\epsilon$) divergence instead of the logarithmic divergence we had in (D.24). This log divergence was important in obtaining the result (D.27), because this log divergence was canceled against the normalization factor in $u_\omega \sim [\log(1/\epsilon)]^{-1/2}$. What we have forgotten is that, if we include the quartic correction $H^{(4)}$, we should also consider corrections to the normalized basis u_ω, which presumably introduces a normalization factor that goes as $\epsilon^{1/2}$. This corrected normalization factor should cancel against the power divergence coming from h^{-2}, thus giving a finite result, which should give (D.27) at the end of the day — namely,

$$\langle R^4 \rangle' \sim \frac{1}{\beta^8 \sqrt{\lambda}}. \tag{D.28}$$

Whatever the modifications due to the quartic term are, the dominant contribution comes from the region $r \sim r_H$ and quantities such as r_c or m cannot enter the final result. Also, the relativistic correction must come with a factor of $\alpha' \sim \lambda^{-1/2}$. There being no other available quantities, $\langle R^4 \rangle'$ must be proportional to (D.28). A fully relativistic formalism in which one can rigorously and explicitly show (D.28) is beyond the scope of the current paper. We leave development of such a formalism for future research.

From (3.35), we have

$$\langle R^2 \rangle = \kappa^{\mathrm{n}}(t=0) \sim \frac{\ell^2}{\alpha' \beta^4} \sim \frac{1}{\beta^4 \sqrt{\lambda}}. \tag{D.29}$$

Therefore, using the formula (D.14) with $\Delta = \beta$, we obtain

$$t_{\mathrm{mfp}} \sim \frac{\beta}{\sqrt{\lambda}} = \frac{1}{T\sqrt{\lambda}}. \tag{D.30}$$

It is satisfactory that this does not depend on the properties of the Brownian particle probe such as m, because t_{mfp} is a time scale associated with the fluid itself.

E Solving equation of motion for general d using matching technique

In this appendix, we solve the wave equation (5.6) for general dimensions using the matching technique for low frequencies $\omega \ll T$. We would like to obtain a solution which is purely ingoing at the horizon and, in particular, determine its behavior near the boundary.

Let us denote by $X_\omega^-(r)$ this particular solution of the wave equation (5.6) which obeys the purely ingoing boundary condition at the horizon $r = r_H$. To determine it, let us consider three regions: (A) a near horizon region with $r \sim r_H$ and $V(r) \ll \omega^2$, (B) an intermediate region with $V(r) \gg \omega^2$, and (C) an asymptotic region with $r \gg r_H$. The idea is to consider the approximate solutions in each of the three regions, and to match these to each other. For more details, see [58] and references therein. As before, we define

$$\rho \equiv \frac{r}{r_H}, \qquad \nu \equiv \frac{\ell^2 \omega}{r_H}. \tag{E.1}$$

In terms of these parameters the constraints on the different regions under consideration, $V(r) \ll \omega^2$ and $V(r) \gg \omega^2$ respectively translate to $\rho - 1 \ll \nu^2$ and $\rho - 1 \gg \nu^2$. Furthermore, the low frequency condition, $\omega \ll T$, can be written as $\nu \ll 1$.

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In region A, where $\rho - 1 \ll \nu^2$ and we can drop the potential $V(r)$ from (5.6), the linearly independent solutions are

$$X_A^{\pm}(r) = e^{\pm i\,\omega\, r_*} \sim \exp\left[\pm \frac{i\,\nu}{d-1} \log(\rho - 1)\right]. \tag{E.2}$$

The purely ingoing solution is $X_A^-(r) = e^{-i\,\omega\, r_*}$. Now, since $\nu \ll 1$ we can focus on a region slightly away from the horizon (still remaining in region A), such that $\exp(-\frac{\nu}{d-1}) \ll \rho - 1 \ll \nu^2$. Here we can approximate the purely ingoing solution as

$$X_A^-(\rho) \sim 1 - \frac{i\,\nu}{d-1} \log(\rho - 1). \tag{E.3}$$

In the asymptotic region C, where $\rho \gg 1$, we can approximate $h(r) \sim 1$. The linearly independent solutions of (5.4) are then

$$X_C^{\pm}(\rho) = \left(1 \mp \frac{i\nu}{\rho}\right) e^{\pm i\,\nu/\rho}. \tag{E.4}$$

The general solution can be written as

$$X_C = C^+\, X_C^+ + C^-\, X_C^-\,, \tag{E.5}$$

which, for $\rho \gg 1$, can be expanded as

$$X_C = (C^+ + C^-)\left(1 + \frac{\nu^2}{2\rho^2} + \dots\right) + (C^+ - C^-)\left(\frac{i}{3}\frac{\nu^3}{\rho^3} + \dots\right). \tag{E.6}$$

Finally, in region B, where $\rho - 1 \gg \nu^2$ and we can drop ω^2 from (5.6), leading then to the general solution

$$X_B(\rho) = B_1 + B_2 \int_{\infty}^{\rho} \frac{d\rho'}{\rho'^4 h(\rho')}, \tag{E.7}$$

where B_1 and B_2 are two integration constants. For $\rho \sim 1$ (but still $\rho - 1 \gg \nu^2$), we can approximate $h(\rho) \sim (d-1)(\rho - 1)$ and (E.7) gives

$$X_B(\rho) = B_1 + B_2 \left[\frac{1}{d-1} \log(\rho - 1) + b\right], \tag{E.8}$$

where b is a constant independent of ν whose precise value is not relevant for our purpose.

We now have the solutions in the three regions A-C; by matching them across the domains of overlap we can relate the various constants of integration. To begin with we determine B_1 and B_2 by matching (E.8) in region B with the solution (E.3) in region A, obtaining

$$B_1 = 1 + i\,b\,\nu, \qquad B_2 = -\,i\,\nu. \tag{E.9}$$

To determine $C^{\pm}$ we expend the solution in region B (E.7) for $\rho \gg 1$ and match it to that in region C (E.6) leading to

$$B_1 = C^+ + C^-, \qquad B_2 = -i\nu^3 (C^+ - C^-). \tag{E.10}$$

It must be borne in mind that we have performed the matching only in the small frequency limit $\nu \ll 1$ and as a result should trust the expressions only at the leading order in ν. Solving (E.9) and (E.10), we finally find that the purely ingoing solution behaves at large ρ as

$$X_\omega^-(\rho) = C^+\, X_C^+ + C^-\, X_C^-, \qquad C^{\pm} = \frac{1}{2}\left(1 \pm \frac{1}{\nu^2} + i\,b\,\nu\right). \tag{E.11}$$

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